教育部「5G行動寬頻人才培育跨校教學聯盟計畫」
5G行動網路協定與核網技術聯盟中心

課程: 5G系統層模擬技術

第二週：MIMO及OFDM技術介紹

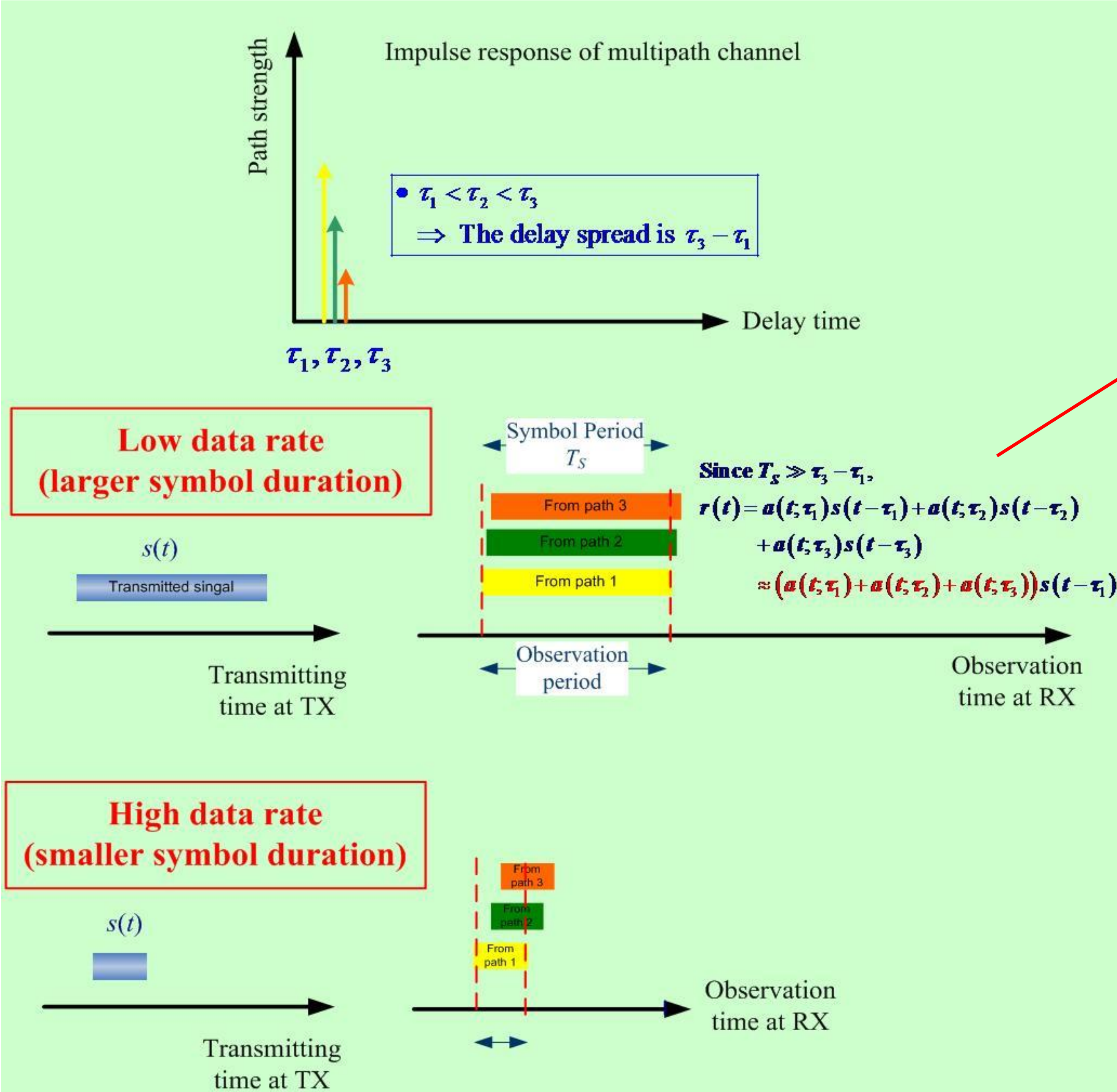
國立中正大學
National Chung Cheng University

Outline

Outline

2.1.1 Introduction(1/4)

- **The demand for high-date-rate multimedia wireless communications (mobile wireless broadband) is growing at an extremely rapid pace.**
  - **2G ➔ WiFi, 3G ➔ WiMAX, LTE ➔ 4G**
- **The "high transmission rate" means:**
  - **Much smaller symbol duration T_s**
  - **More serious inter-symbol interference (ISI) due to multipath**
  - **The number of taps L required for an equalizer is typically large**
  - **These equalizers are quite complex.**
- **The main advantage of OFDM is to provide high speed transmission without using complex equalizers at receiver.**



For large symbol duration, the ISI is effectively eliminated.

(ISI from multipath)

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2.1.1 Introduction(2/4)

- Starting around 1990, OFDM has been adopted in commercial systems such as:
  - Broadband wired access (ASDL)
  - Digital audio and video broadcasting in Europe (DAB, DVB)
  - Wireless LANs (IEEE 802.11a, HIPERLAN-2)
  - Wireless MAN (IEEE 802.16e)
  - HDTV
  - One of two competing proposals for the IEEE 802.15 ultrawideband radios

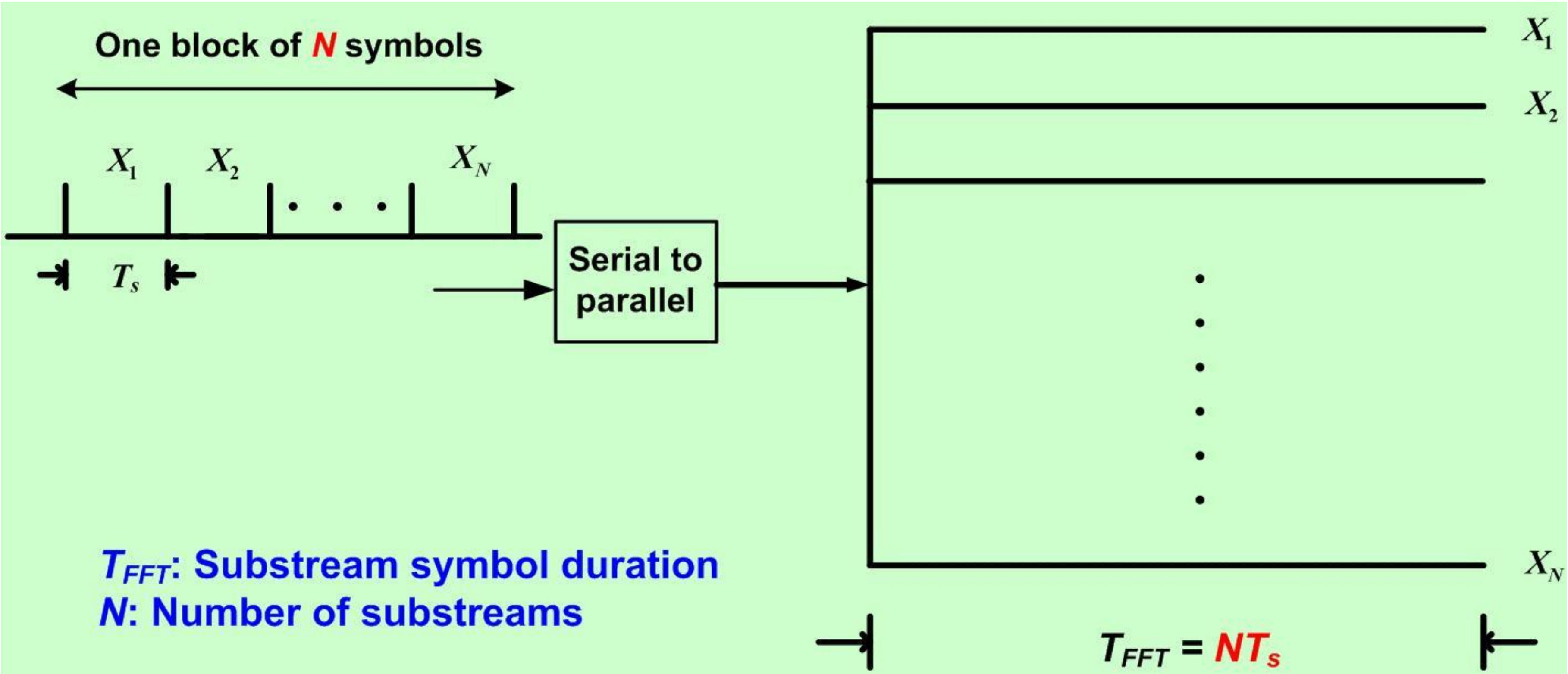
2.1.1 Introduction(3/4)

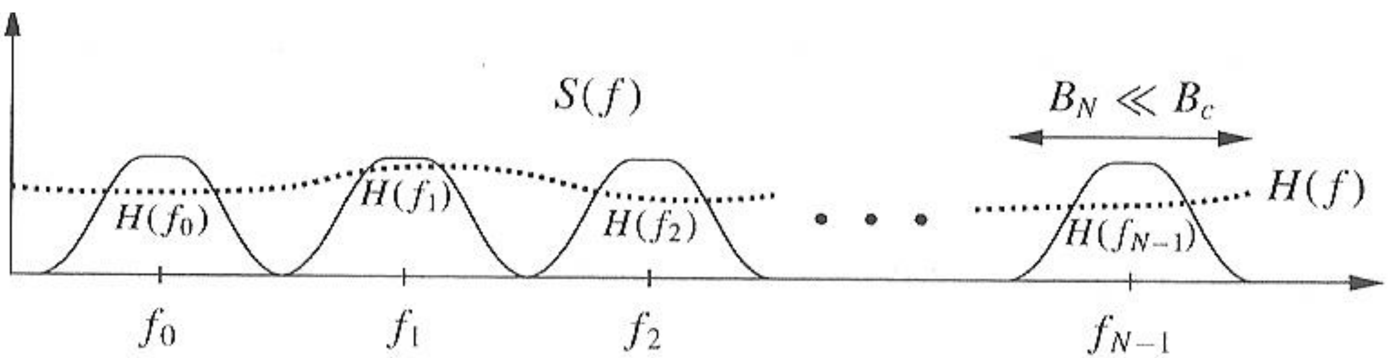
- **OFDM Is a Multicarrier Modulation Scheme:**
  - **The basic idea of multicarrier modulation is to split the high data-rate stream into a number of substreams that are transmitted in parallel over different subchannels.**

    ➔ Block transmission
  - **The data rate on each of the subchannels is much less than the input date rate.**
  - **The number of substreams is chosen to ensure that:**
    - **The symbol time on each substream is much greater than the delay spread of the channel.**
    - **Or, equivalently, each subchannel bandwidth is less than the coherence bandwidth of the channel.**

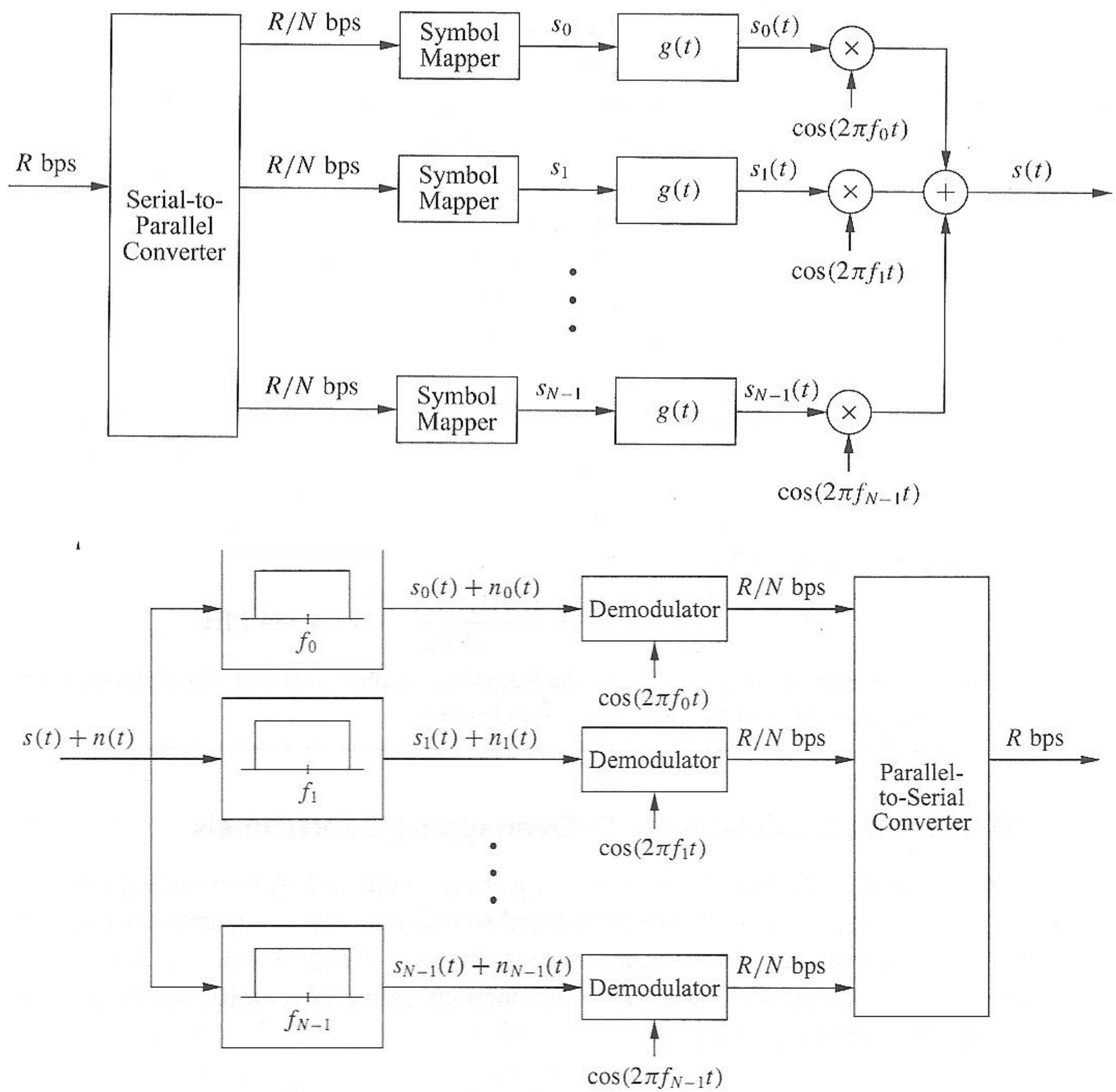
  ➔ The subchannels experience relatively flat fading.

  ➔ The ISI (inter-symbol interference) on each subchannel is much small.





(Illustration of multicarrier modulation)

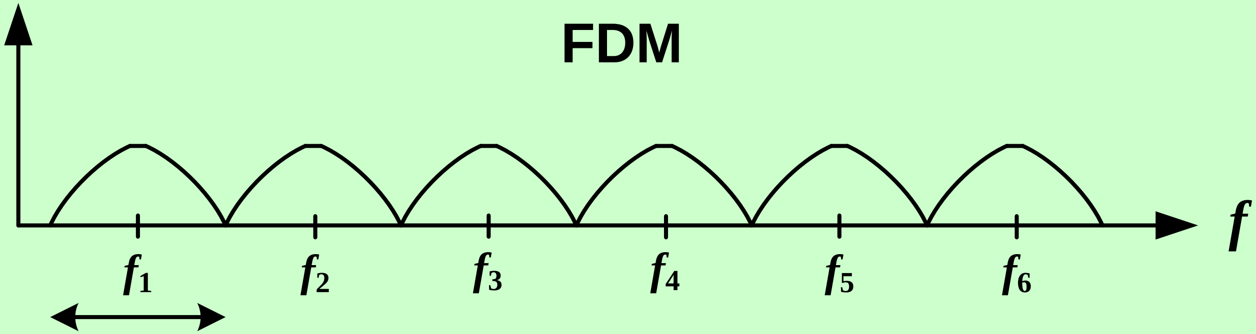


(Multicarrier transceiver)

2.1.1 Introduction(4/4)

- OFDM vs. FDM (Frequency-Division Multiplexing)
  - The main difference between FDM and OFDM is that in OFDM, the spectra of the individual subcarriers overlap.
    - The OFDM subcarriers are orthogonal so that they can be separated out by the receiver.
  - That is, in OFDM, the spectral efficiency is improved by overlapping the subcarriers.

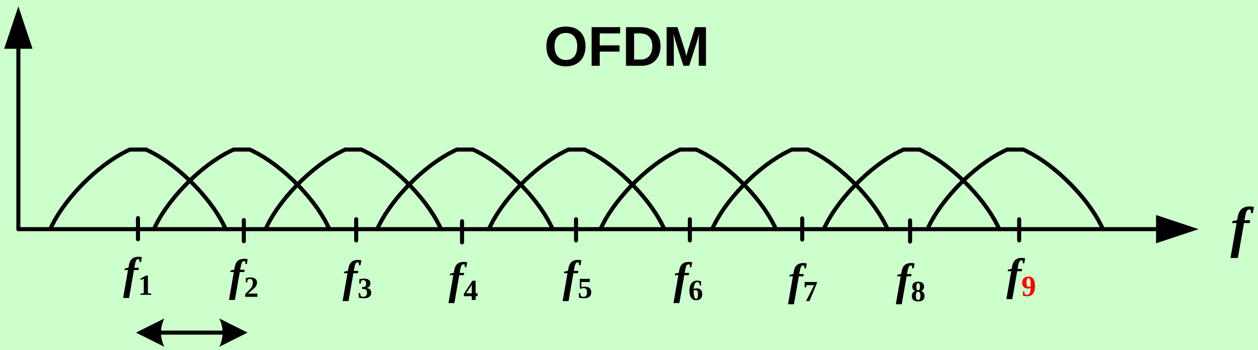
FDM



$(1+\beta)/T_s$ where $\beta \le 1$ is the rolloff factor of raise-cosine shaping function.

$\longleftarrow \quad W \quad \longrightarrow$

OFDM



Δf : subcarrier spacing

$T_{FFT} = 9T_s$

$$\longleftarrow \quad W = (9-1)\Delta f + (1+\beta)/T_{FFT} \quad \longrightarrow$$

In FDM, *W*-Hz bandwidth accommodates 6 carriers, whereas 9 carriers for OFDM.

2.1.2 Minimum Subcarrier Spacing for Orthogonal Subcarriers(1/2)

- **The spectral efficiency of OFDM is improved by overlapping the subchannels.**
  - **Subcarriers are orthogonal in time, but overlapped in frequency.**
- **The minimum "adjacent subcarrier spacing Δf" required to maintain subcarrier orthogonality in time domain:**

$$\textbf{Orthogonal: } \int_{\Delta}^{\Delta+T_{FFT}} \cos\left(2\pi f_i t+\theta_i\right)\cos\left(2\pi f_j t+\theta_j\right)dt=0,\quad f_k=k\Delta f,\ k=1,2,\ldots,N$$

$$\textbf{where } \Delta f=\begin{cases}1/\left(2T_{FFT}\right) & \textbf{for coherent systems } \left(\theta_i=\theta_j=0\right)\\ 1/T_{FFT} & \textbf{for non-coherent systems } \left(\textbf{otherwise, } \theta_i\neq\theta_j \textbf{ or } \theta_i=\theta_j\neq 0\right)\end{cases}$$

$$\left(T_{FFT}=NT_s\textbf{: substream-symbol duration}\right)$$

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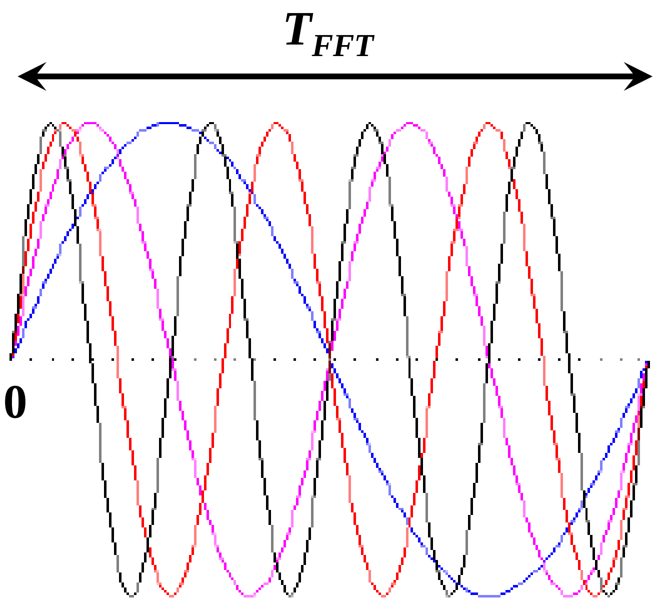
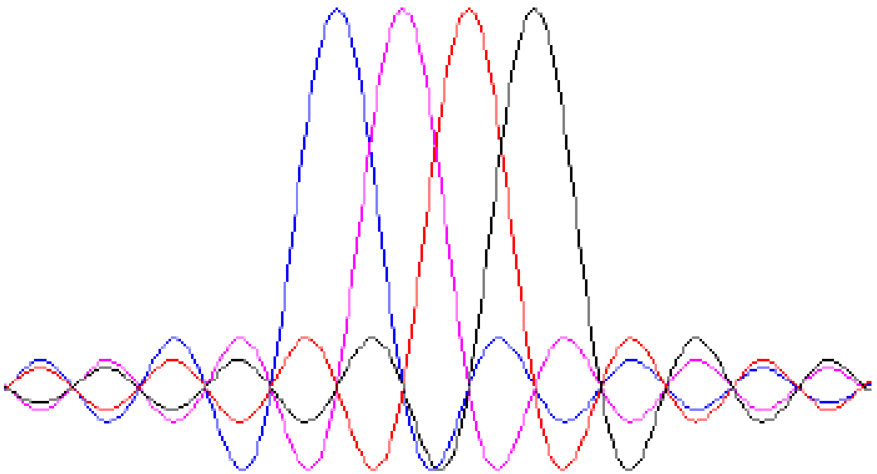
2.1.2 Minimum Subcarrier Spacing for Orthogonal Subcarriers(2/2)



Orthogonal in Time-domain



Overlapped in Frequency-domain

Orthogonal in time domain:

$$\int_0^{T_{FFT}} \cos(2\pi f_i t)\cos(2\pi f_j t)\,dt = 0 \text{ for } i \neq j, \quad \text{where } f_k = k\Delta f \text{ and } \Delta f = 1/T_{FFT}$$

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2.1.3 Spectral Efficiency Comparison between Multicarrier and Single-carrier Modulations

- **Single-carrier Modulation**
  - **The required transmission bandwidth: $W = (1+\alpha)/T_s$**
    (T_s: symbol duration; $\alpha \leq 1$: rolloff factor)
  - **The transmission bit rate: $R = (\log_2 M)/T_s$ (bits/s)**
    (M: Alphabet size, i.e., number of bits conveying in one symbol)
  - **Spectral efficiency ➔ $R/W = (\log_2 M)/(1+\alpha)$ (bits/s/Hz)**
- **Multicarrier Modulation with Overlapping Subchannels**
  - **$W = (N+\alpha)\,\Delta f = (N+\alpha)/(NT_s)$**
  - **$R = (\log_2 M)/T_s$ (bits/s)**
  - **Spectral efficiency ➔ $R/W = N \cdot \log_2 M/(N+\alpha)$ (bits/s/Hz)**

    ➔ When $N \gg \alpha$, $R/W \approx \log_2 M$ (bits/s/Hz)
  - **$(1+\alpha)$ more efficient than signal-carrier case**

2.1.4 Adaptive Loading

- **Vary the data rate R_i and power P_i, assigned to each subchannel i, according the associated subchannel condition.**

  ➔ **Water-filling algorithm:**

$$C = \max_{P_i : \sum_i P_i = P} \sum_{i=0}^{N-1} R_i\left(\alpha_i, P_i, N_0\right) = \max_{P_i : \sum_i P_i = P} \sum_{i=0}^{N-1} B \log_2\left(1 + \alpha_i^2 P_i / N_0 B\right) \quad \text{(bits/sec)}$$

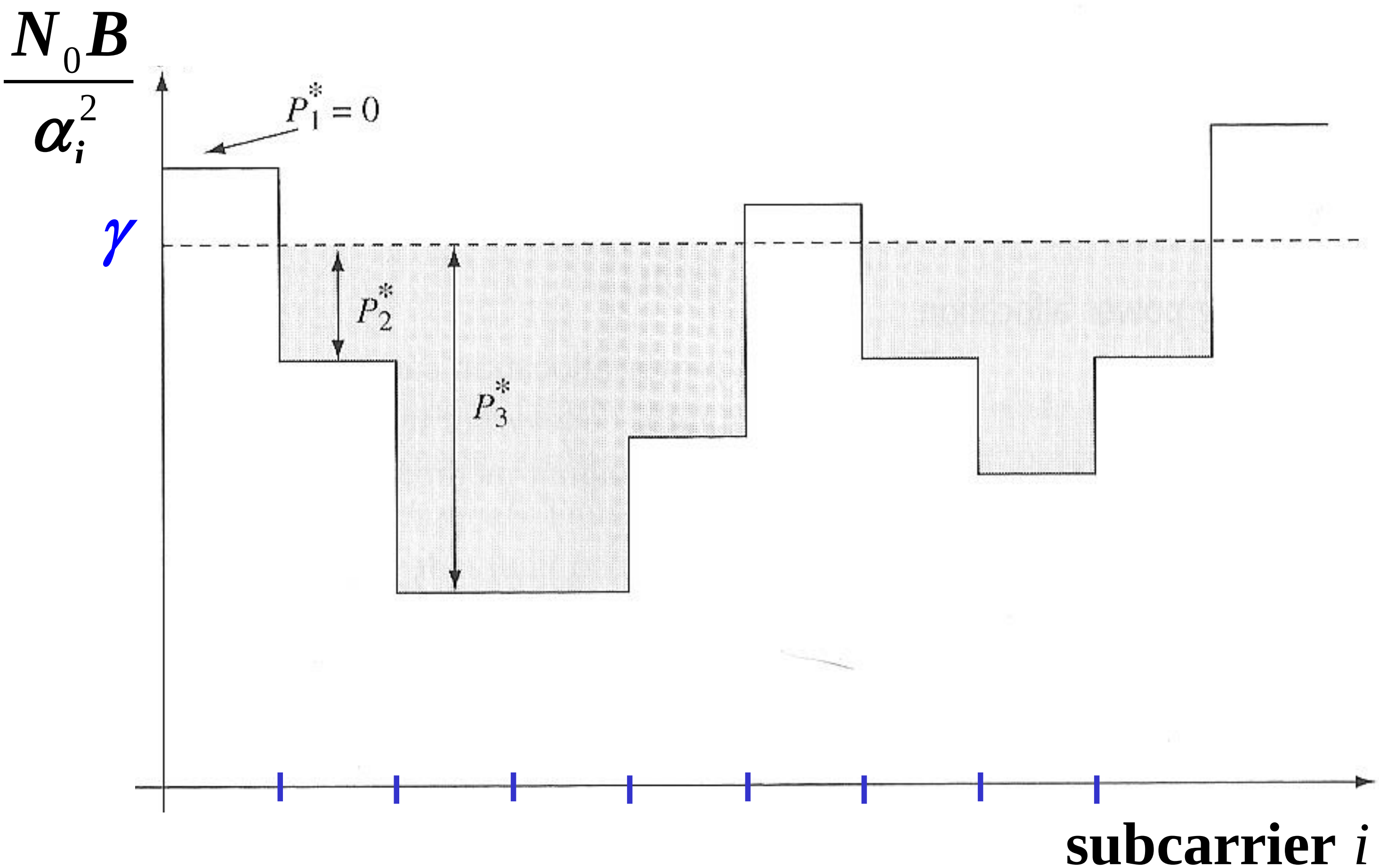
- N subchannels, each of which with bandwidth B; i-th subchannel gain α_i
- P : total power constraint; P_i: power allocation to subchannel i
- N_0 : AWGN power spectral density

The power allocation P_i that maximizes the total data rate C is:

$$P_i = \begin{cases} \gamma - N_0 B / \alpha_i^2, & N_0 B / \alpha_i^2 \le \gamma \\ 0, & N_0 B / \alpha_i^2 > \gamma \end{cases}$$

γ: a cutoff value such that $\sum_{i=0}^{N-1} P_i = P$

Note that the water-filling algrithm requires the knowldege of subchannel fadings $\{\alpha_i\}$


$\frac{N_0 B}{\alpha_i^2}$
$P_1^* = 0$
γ
P_2^*
P_3^*
subcarrier i

2.1.5 Direct Method of Generating OFDM Signals

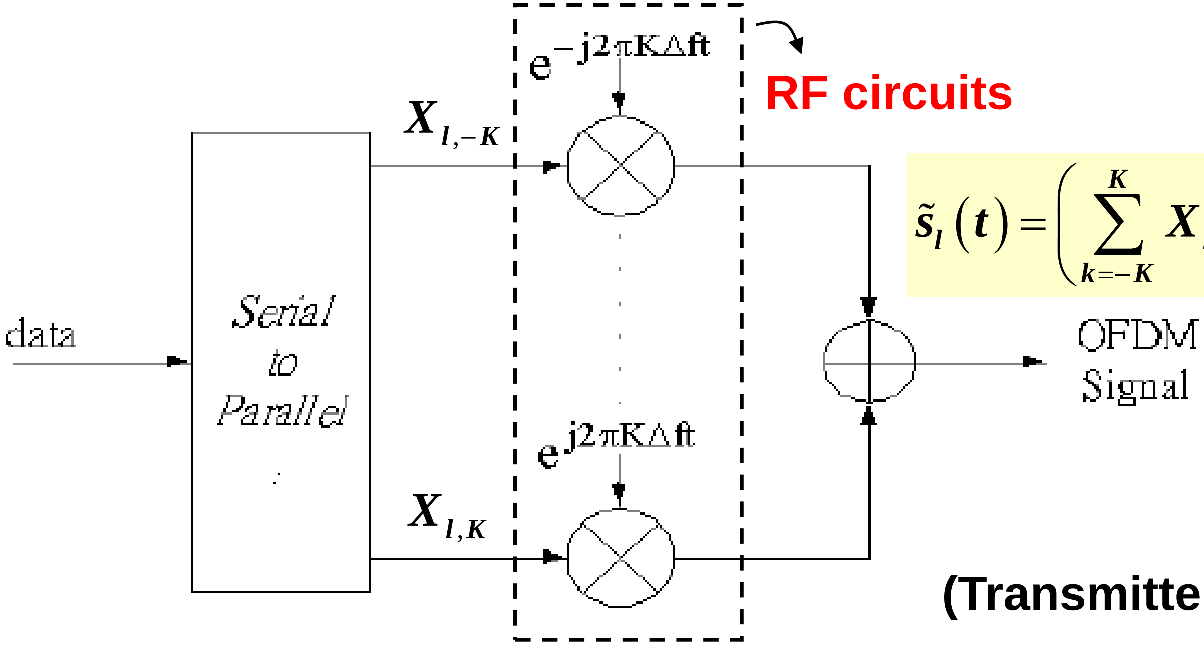
- **Use N sets of RF modules to generate OFDM signals:**

$$\boldsymbol{s}(\boldsymbol{t}) = \sum_{l}\left[\sum_{k=-K}^{K}\left\{\mathrm{Re}\left\{X_{l,k}\right\}\cos\left(2\pi\left(f_c+k\Delta f\right)t\right)-\mathrm{Im}\left\{X_{l,k}\right\}\sin\left(2\pi\left(f_c+k\Delta f\right)t\right)\right\}\right]U\left(t-lT_{FFT}\right),$$

$$= \sum_{l}\mathrm{Re}\left\{\left(\sum_{k=-K}^{K}X_{l,k}e^{j2\pi k\Delta f t}\right)e^{j2\pi f_c t}\right\}U\left(t-lT_{FFT}\right)$$

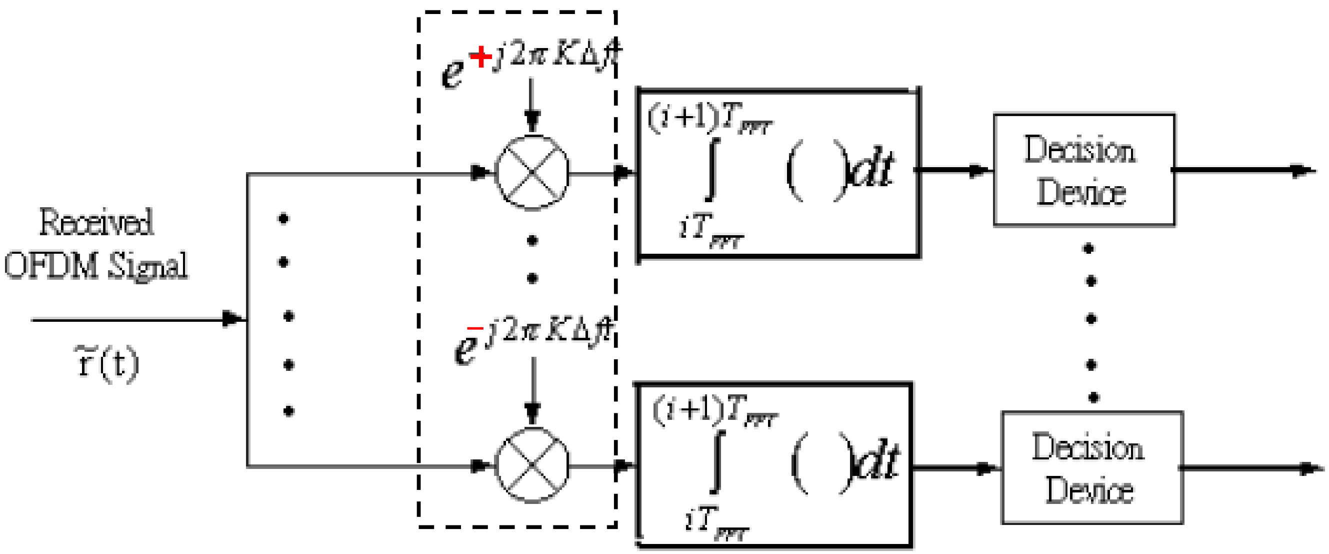
$$= \sum_{l}\mathrm{Re}\left\{\tilde{s}_l\left(t\right)e^{j2\pi f_c t}\right\}$$

- l : **OFDM signal index;** k : **Subcarrier index**
- $X_{l,k}$: **The complex data carried on k-th subcarrier of l-th OFDM signal**
- $2K+1$: **Number of data subcarriers (including 1 DC subcarrier);** $U\left(t\right)=\begin{cases}1, & 0\le t\le T_{FFT}\\ 0, & \textbf{otherwise}\end{cases}$
- $\tilde{s}_l\left(t\right)=\left(\sum_{k=-K}^{K}X_{l,k}e^{j2\pi k\Delta f t}\right)U\left(t-lT_{FFT}\right)$: **Equivalent lowpass signal**



$$\tilde{s}_l(t) = \left(\sum_{k=-K}^{K} X_{l,k} e^{j2\pi k\Delta ft} \right) U(t - lT_{FFT})$$

(Transmitter)



(Receiver)

2.1.6 Inter-carrier Interference (ICI)

- **Synchronization error (including both frequency offset f_e and timing error τ_e) will cause inter-carrier interference (ICI):**

To extract data $X_{l,m}$ on m-th subcarrier of l-th OFDM signal

$$\frac{1}{T_{FFT}}\int_{lT_{FFT}}^{(l+1)T_{FFT}+\tau_e}\tilde{s}_l(t)e^{-j2\pi(m\Delta f+f_e)t}dt$$

$$\tilde{s}_l(t)=\left(\sum_{k=-K}^{K}X_{l,k}e^{j2\pi k\Delta f t}\right)U(t-lT_{FFT})$$

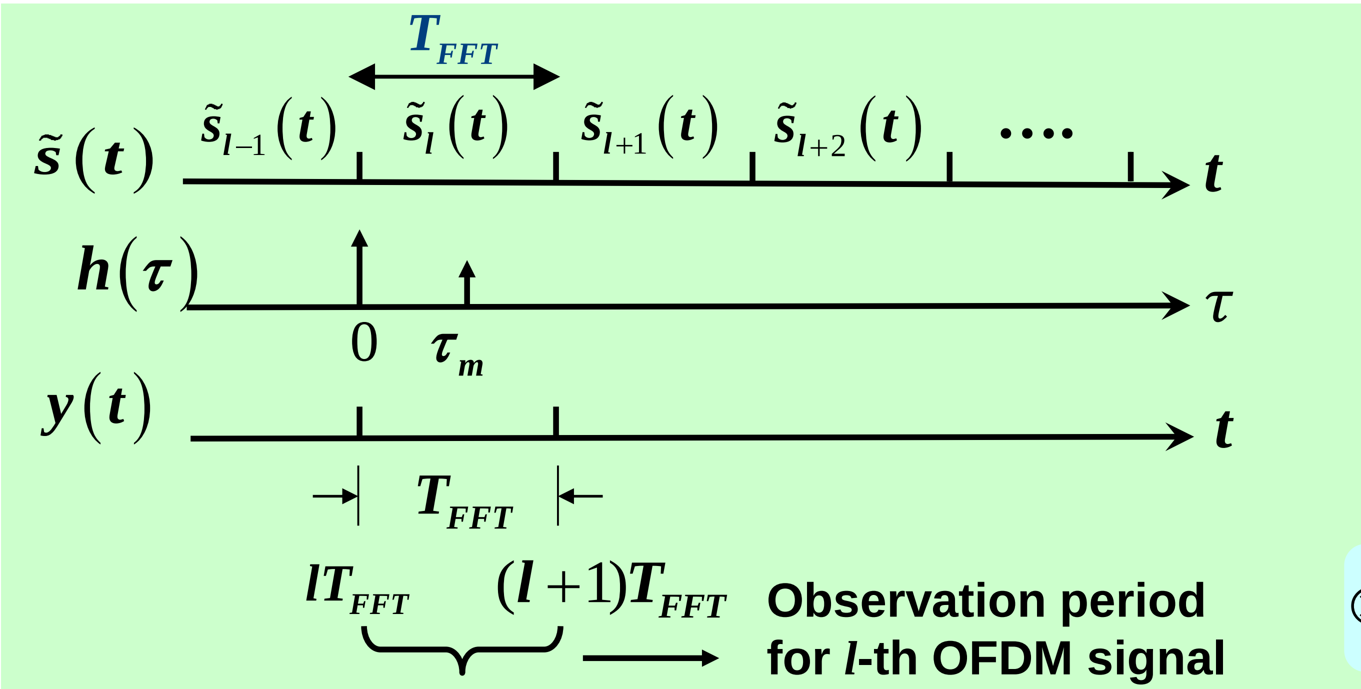
$$=\frac{1}{T_{FFT}}\sum_{k=-K}^{K}X_{l,k}\int_{lT_{FFT}}^{(l+1)T_{FFT}+\tau_e}e^{j2\pi((k-m)\Delta f+f_e)t}dt$$

$$\frac{1}{T_{FFT}}\int_{0}^{T_{FFT}}e^{j2\pi(k-m)\Delta f t}dt=\begin{cases}1, & k=m\\0, & k\neq m\end{cases},\quad \Delta f=\frac{1}{T_{FFT}}$$

$$=\begin{cases}X_{l,m}, & \text{perfect synch. } \tau_e=f_e=0\\ \underbrace{X_{l,m}\left(\frac{1}{T_{FFT}}\int_{lT_{FFT}}^{(l+1)T_{FFT}+\tau_e}e^{j2\pi f_e t}dt\right)}_{\text{signal part}}+\underbrace{\left(\sum_{k\neq m}X_{l,k}\frac{1}{T_{FFT}}\int_{lT_{FFT}}^{(l+1)T_{FFT}+\tau_e}e^{j2\pi((k-m)\Delta f+f_e)t}dt\right)}_{\text{ICI part due to synch. error}}, & \tau_e\neq 0 \text{ or } f_e\neq 0\end{cases}$$

2.1.7 Inter-block Interference (IBI)

- **Multipath (delay spread) will cause inter-block interference (IBI):**



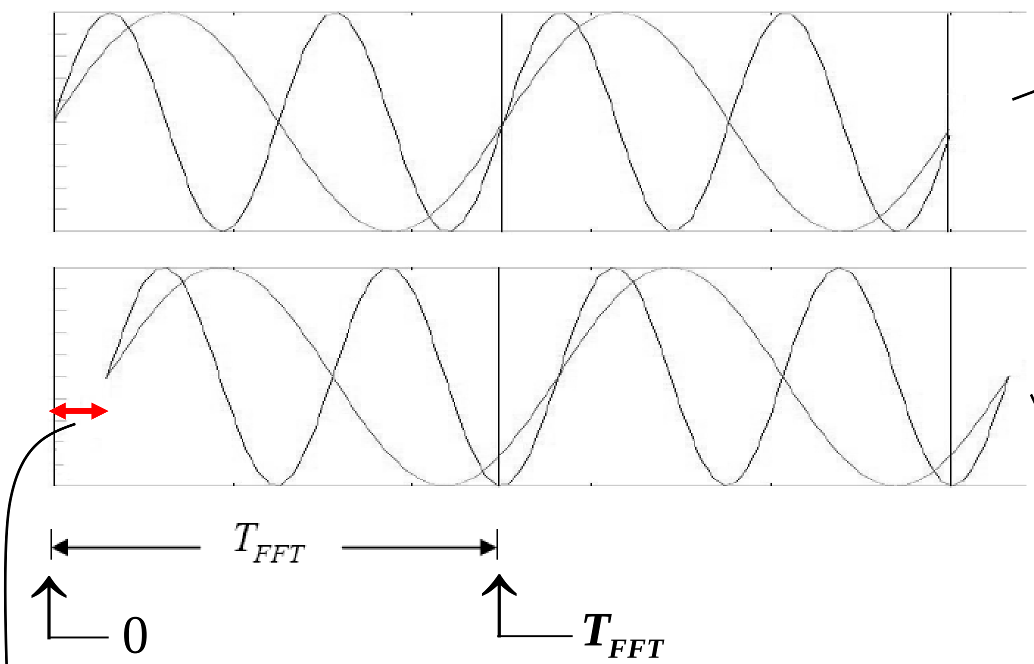
$\otimes$: convolution

$$y(t)\Big|_{t=lT_{FFT}}^{(l+1)T_{FFT}} = \left(\sum_{l'} \tilde{s}_{l'}(t)\right) \otimes h(t)\Big|_{t=lT_{FFT}}^{(l+1)T_{FFT}} = \left(\tilde{s}_{l-1}(t) + \tilde{s}_l(t)\right) \otimes h(t)\Big|_{t=lT_{FFT}}^{(l+1)T_{FFT}}$$

$$= \underbrace{\tilde{s}_{l-1}(t) \otimes h(t)\Big|_{t=lT_{FFT}}^{lT_{FFT}+\tau_m}}_{\text{IBI due to multipath}} + \underbrace{\tilde{s}_l(t) \otimes h(t)\Big|_{t=lT_{FFT}}^{(l+1)T_{FFT}}}_{\text{signal part}}$$

The signal part will contain ICI due to delay spread.

ICI due to Multipath



Observation duration

Reflection delay, τ_m

First arriving path

$\tilde{s}_l(t)U(t)$

➔ Signals from These Two Paths Are Not Orthogonal Any More due to the Incompleteness of the Reflected Waveform

From other delayed path

$\tilde{s}_l(t-\tau_m)U(t-\tau_m)$

Not Orthogonal in Time Domain:

$$\int_{\tau_m}^{T_{FFT}} \cos(2\pi f_i t)\cos(2\pi f_j t)\,dt \neq 0, \quad f_k = k\Delta f$$

$\rightarrow \because$ **Integration over partial period**

ICI due to Multipath (Cont.)

$$\tilde{s}_l(t) = \left(\sum_{k=-K}^{K} X_{l,k} e^{j2\pi k \Delta f t} \right) U(t - lT_{FFT})$$

$\rightarrow$ **To extract $X_{l,m}$ on subcarrier m:**

$$\frac{1}{T_{FFT}} \int_{lT_{FFT}}^{(l+1)T_{FFT}} \left(\underbrace{\tilde{s}_l(t) U(t - lT_{FFT})}_{\text{first arrival}} + \underbrace{\tilde{s}_l(t-\tau_m) U(t - \tau_m - lT_{FFT})}_{\text{reflection wave}} \right) e^{-j2\pi m \Delta f t} dt$$

$$= \sum_{k=-K}^{K} X_{l,k} \int_{lT_{FFT}}^{(l+1)T_{FFT}} e^{j2\pi(k-m)\Delta f t} dt + \frac{1}{T_{FFT}} \sum_{k=-K}^{K} X_{l,k} e^{-j2\pi k \Delta f \tau_m} \int_{lT_{FFT}+\tau_m}^{(l+1)T_{FFT}} e^{j2\pi(k-m)\Delta f t} dt$$

$$= X_{l,m} + \underbrace{\frac{1}{T_{FFT}} \sum_{k=-K}^{K} X_{l,k} e^{-j2\pi k \Delta f \tau_m} \int_{lT_{FFT}+\tau_m}^{(l+1)T_{FFT}} e^{j2\pi(k-m)\Delta f t} dt}_{\text{ICI from "incomplete" OFDM symbol due to multipath}}$$

➔ **How can we remove both IBI and ICI caused by multipath?**

➔ **Utilizing guard interval (or cyclic prefix) can effectively eliminate IBI and maintain orthogonality among subcarriers.**

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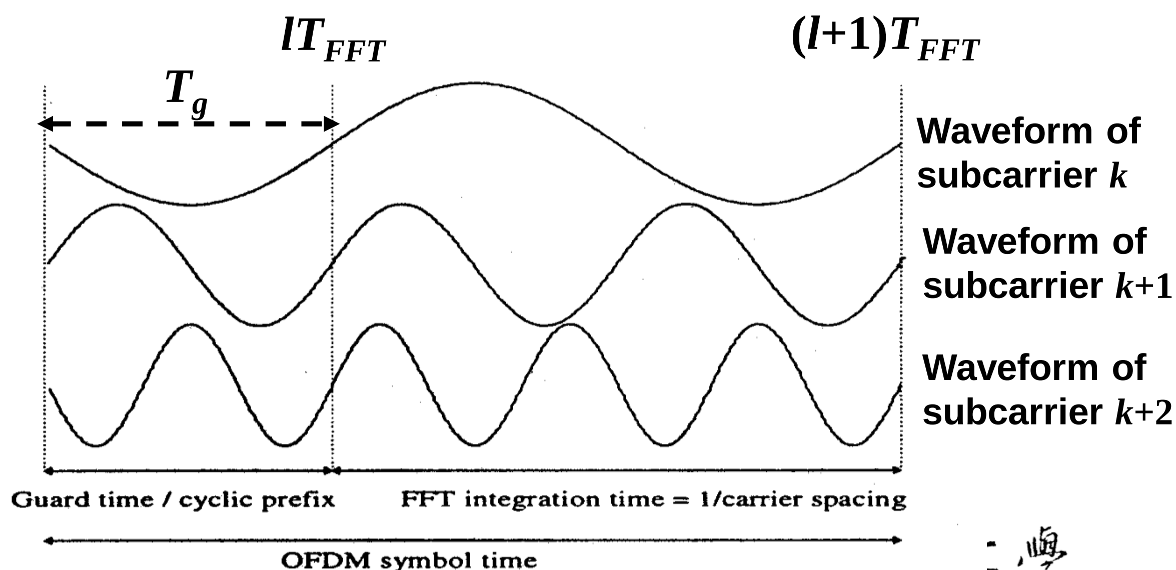
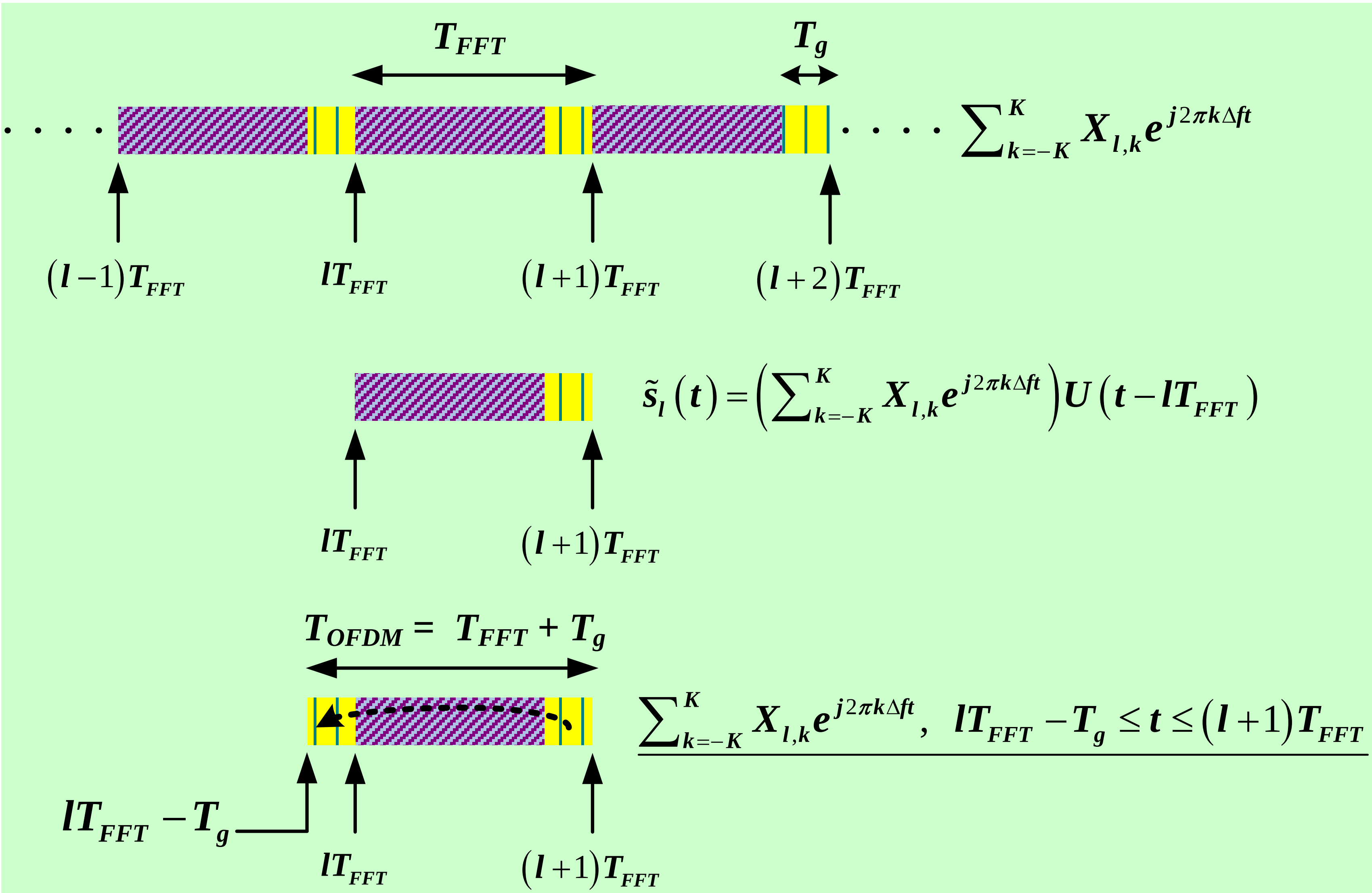
2.1.8 Guard Interval (or Cyclic Prefix) (1/2)

- **Observation on** $\tilde{s}_l(t) = \left(\sum_{k=-K}^{K} X_{l,k} e^{j2\pi k \Delta f t}\right), \quad lT_{FFT} < t \le (l+1)T_{FFT}$
  - **"**$\left\{X_{l,k} e^{j2\pi k \Delta f t}\right\}_k$**" is periodic with the fundamental frequency and period, $\Delta f = GCD\{k\Delta f, k = -K, \ldots, K\}$ and $T_{FFT} = 1/\Delta f$, respectively.**
  - **How to add a prefix of length T_g to the beginning of the signal $\tilde{s}_l(t)$, such that over $(lT_{FFT} - T_g, (l+1)T_{FFT}]$, any T_{FFT}-length segment of the resultant signal constitutes a full period of the signal?**

By appending the last segment of T_g duration of $\tilde{s}_l(t)$ to the beginning of itself.

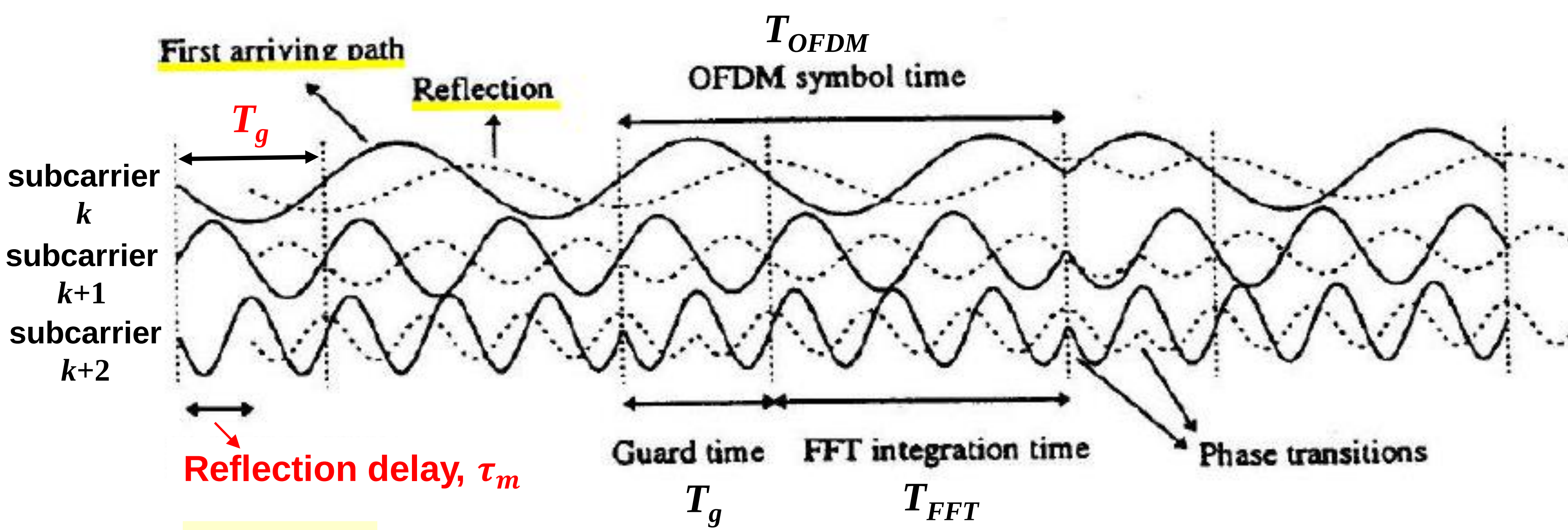


: The last segment of T_g duration of an OFDM symbol



2.1.8 Guard Interval (or Cyclic Prefix) (2/2)

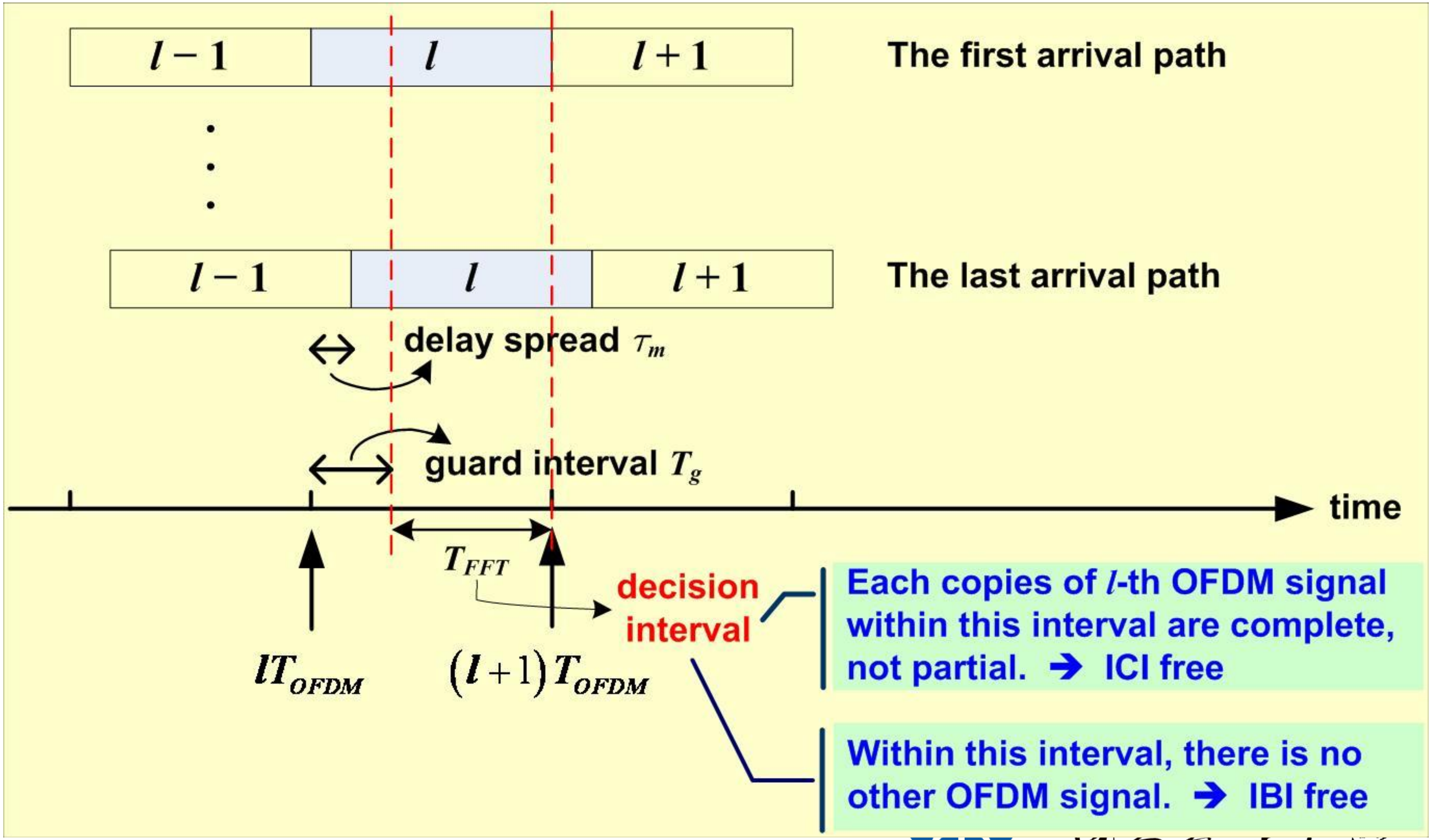
- As long as guard interval T_g > delay spread τ_m, both IBI and ICI caused by multipath can be completely eliminated.



$$\Rightarrow T_g > \tau_m$$

Length of Cyclic Prefix > Delay Spread

➔ Completely Eliminate IBI and ICI Caused by Multipath



2.1.9 OFDM with IFFT/FFT Implementation(1/5)

- **Review Notations**
  - T_s: Original symbol duration from serial input
  - N: Serial-to-parallel size
  - $T_{FFT} = NT_s$: Useful data duration
  - T_g: Guard interval
  - $T_{OFDM} = T_{FFT} + T_g$: OFDM symbol duration
  - $\Delta f = 1/T_{FFT}$: Subcarrier spacing
  - $2K + 1$: Number of data subcarriers (including 1 DC subcarrier)
  - $N_g = T_g/T_s$
  - $N_{OFDM} = T_{OFDM}/T_s = N + N_g$
  - $2W_{base} = (2K +$ ♌$)\,\Delta f$: The passband transmission bandwidth of OFDM

2.1.9 OFDM with IFFT/FFT Implementation(2/5)

- **Generate OFDM signal using IDFT eliminating the necessity of *N* RF modules. ➔ Practical implementation**
- **Sampling of OFDM Signals with Sampling Period T_s**
  - **The required passband transmission bandwidth:**

$$2W_{base} = (2K + \delta)\Delta f,$$

    where W_{base} is the baseband bandwidth.
  - **Equivalence condition:**

$$1/T_s \geq 2W_{base}$$

$$\Rightarrow N \geq 2K + \delta, \quad \delta \leq 1$$

$$\Rightarrow N \geq 2K + 1 \text{ (number of data subcarriers)}$$

  - **In practice, $2K + 1$ should be less than N, in order to have reliable filter (feasible transition band).**

2.1.9 OFDM with IFFT/FFT Implementation(3/5)

- According to the equivalence theorem, we can say the **OFDM samples $\{x_n\}$** with sample period T_s would be equivalent to the **analog OFDM signal** $\tilde{s}(t) = \left(\sum_{k=-K}^{K} X_k e^{j2\pi k \Delta f t}\right) U(t - T_{FFT})$:

$$x_n \doteq \frac{1}{N} \tilde{s}(t)\Big|_{t=nT_s} = \frac{1}{N} \sum_{k=-K}^{K} X_k e^{j2\pi k \Delta f t}\Bigg|_{t=nT_s} = \frac{1}{N} \sum_{k=-K}^{K} X_k e^{j2\pi k \frac{1}{NT_s}(nT_s)}$$

$$= \frac{1}{N} \sum_{k=-K}^{K} X_k e^{j\frac{2\pi kn}{N}}, n = 0, 1, \cdots, N-1$$

The OFDM samples $\{x_n\}$ are periodic with period of N. That is, $x_n = x_{n+N}$.

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2.1.9 OFDM with IFFT/FFT Implementation(4/5)

- **OFDM with IFFT/FFT Implementation**

$$x_n = IDFT\{X_k\} = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi kn}{N}}, n = 0,1,\cdots,N-1$$

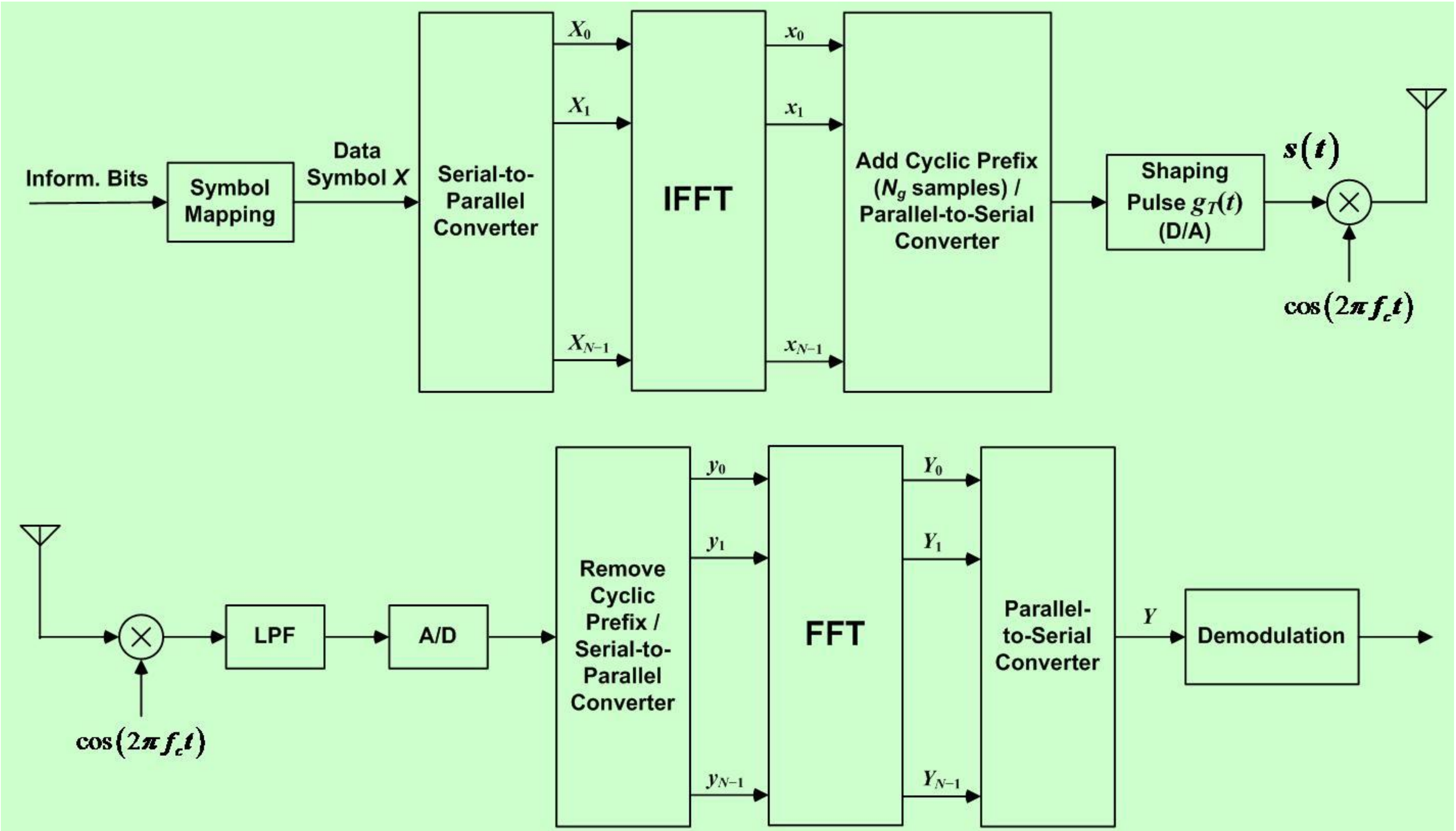
(Time-domain samples)

- **Only 2*K*+1 subcarriers carries data symbols.**
- **The other *N* − (2*K* +1) subcarriers are virtual carriers carrying nothing.**

$$X_k = DFT\{x_n\} = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi kn}{N}}, k = 0,1,\cdots,N-1$$

(Frequency data)

2.1.9 OFDM with IFFT/FFT Implementation(5/5)



OFDM signal: $s(t) = \sum_{l=-\infty}^{\infty} \sum_{n=-N_g}^{N-1} x_{l,n} g_T\left(t - nT_s - lN_{OFDM}T_s\right)$

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2.1.10 Key Advantages and Disadvantages of OFDM(1/2)

- Key Advantages
  - No need for time-domain equalization for high speed transmission
    - Robust against multipath distortions, as channel equalization can easily be performed in the frequency domain through a bank of one-tap multipliers.
  - Higher spectral efficiency
    - Improved by overlapping the subcarriers.
  - Adaptive loading over subchannels
    - Allow independent selection of the modulation parameters (transmit power, constellation size and coding scheme) over each subcarrier.

2.1.10 Key Advantages and Disadvantages of OFDM(2/2)

- **Key Disadvantages**
  - **High peak average power ratio (PAPR)**

$$\mathbf{PAPR} \doteq \frac{\max_n \left|\boldsymbol{x}[\boldsymbol{n}]\right|^2}{\boldsymbol{E}\left[\left|\boldsymbol{x}[\boldsymbol{n}]\right|^2\right]}$$

    - **Stringent linearity requirement**
    - **Complex RF transmitter (large backoff PA (larger linearity region) ➔ high resolution for receiver A/D)**
    - **Low power efficiency**
  - **Difficult to maintain orthogonality in high mobility environments**
    - **Inter-carrier interference**

[1] IEEE 802.11 WORKING GROUP (2003) Draft Supplement to STANDARD FOR Telecommunications and Information Exchange Between Systems-LAN/MAN Specific Requirements-Part 11： Wireless Medium Access Control (MAC) and Physical Layer (PHY)specifications

Data rate (Mbits/s)	Modulation	Coding rate (R)	Coded bits per subcarrier (N_{BPSC})	Coded bits per OFDM symbol (N_{CBPS})	Data bits per OFDM symbol (N_{DBPS})
6	BPSK	1/2	1	48	24
9	BPSK	3/4	1	48	36
12	QPSK	1/2	2	96	48
18	QPSK	3/4	2	96	72
24	16-QAM	1/2	4	192	96
36	16-QAM	3/4	4	192	144
48	64-QAM	2/3	6	288	192
54	64-QAM	3/4	6	288	216

$$\begin{aligned} \boldsymbol{R}_{\min} &= 48 \textbf{ data subcarriers} \cdot \frac{1 \textbf{ coded bit}}{\textbf{subcarrier symbol}} \cdot \frac{1/2 \textbf{ data bit}}{1 \textbf{ coded bits}} \cdot \frac{\mathbf{20 \cdot 10^6 subcarrier\ symbol}}{\textbf{seconds}} \\ &= 6 \textbf{ Mbps} \end{aligned}$$

$$\begin{aligned} \boldsymbol{R}_{\max} &= 48 \textbf{ data subcarriers} \cdot \frac{6 \textbf{ coded bit}}{\textbf{subcarrier symbol}} \cdot \frac{3/4 \textbf{ data bit}}{1 \textbf{ coded bits}} \cdot \frac{\mathbf{20 \cdot 10^6 subcarrier\ symbol}}{\textbf{seconds}} \\ &= 54 \textbf{ Mbps} \end{aligned}$$

Outline

2.2.1 Introduction(1/4)

- Demands for capacity in wireless communications have been rapidly increasing worldwide.
- However, the available radio spectrum is limited (scarce).
- On the other hand, advances in error control coding (ECC), such as turbo code and LDPC code, made it feasible to approach the Shannon capacity limit for systems with a single antenna link.
- Although ECC can achieve a coding gain, but suffers a loss in bandwidth due to code redundancy.
- Advances in "*vastly*" improving spectral efficiency are available through increasing the number antennas at both the transmitter and receiver.

2.2.1 Introduction(2/4)

- In this chapter, we derive fundamental capacity limits for transmission over multiple-input multiple-output (MIMO) channels.
  - They are mainly based on the theoretical work developed by Telatar [2] and Foschini [3].
- In later chapters, we consider some practical coding techniques which potentially approach the derived capacity limits.
  - For example, BLAST coding technique [4] can attain the spectral efficiencies up to 42 bits/sec/Hz.
  - This represents a spectacular increase compared to currently achievable spectral efficiencies of 2~3 bits/sec/Hz, in cellular mobile and wireless LAN systems.

2.2.1 Introduction(3/4)

- In this course, individual channels between given pairs of transmit and receive antennas are modeled by an *independent flat* Rayleigh fading process.
- That is, we limit the analysis to the case of narrowband channels, so that they can be described by frequency flat models.
- The results are generalized to wideband channels, simply by considering a wideband channel as a set of orthogonal narrow-band channels. (OFDM wideband systems)

2.2.1 Introduction(4/4)

- The *independent* Rayleigh fading model can be approximated in MIMO channels where antenna element spacing is considerably larger than the carrier wavelength, or the incoming wave "incidence angle spread" is relatively large (>30°).
  - Note that: there have been many measurements indicating that if two receive antennas are used to provide diversity at the base station receiver, they must be on the order of ten wavelengths apart to provide sufficient de-correlation.

2.2.2 MIMO System Model(1/11)

- Consider a point-to-point MIMO system with n_T transmit and n_R receive antennas.
- The transmitted signals in each period are represented by an $n_T \times 1$ column vector x.
  - $\mathrm{x} = [x_1, x_2, \ldots, x_{nT}]^T$, where the ith component x_i, refers to the transmitted signal from antenna i.
- The channel is described by an $n_R \times n_T$ channel matrix, H.
  - The (i, j)-th component of H, denoted by h_{ij}, represents the fading coefficient (with complex-Gaussian distribution) "from the jth transmit antenna to the ith receive antenna".
- For i.i.d. Rayleigh fading channels, we have

$$E\left\{h_{ij} \cdot h_{kl}^*\right\} = \begin{cases} E\left\{\left|h_{ij}\right|^2\right\} = 1, & \text{for } i = k,\ j = l \\ E\left\{h_{ij}\right\} E\left\{h_{kl}^*\right\} = 0, & \text{otherwise} \end{cases}.$$

$$n_T = 3, n_R = 2 \rightarrow$$
$$\mathbf{x} = \left[x_1, x_2, x_3\right]^T$$
$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \end{bmatrix}$$

2.2.2 MIMO System Model(2/11)

- The received signals are represented by an $n_R \times 1$ column vector, r.
  - $\mathbf{r} = [r_1, r_2, \ldots, r_{n_R}]^T = \mathbf{Hx} + \mathbf{n}$, where the ith component r_i, refers to the received signal at antenna i.
- Point-to-Point MIMO

$$\boxed{\textbf{noise vector} \rightarrow \mathbf{n} = \left[n_1, n_2, \cdots, n_{n_R}\right]^T}$$

$$r_i = \sum_{j=1}^{n_T} h_{ij} x_j + n_i, \ \ i = 1, 2, \ldots, n_R$$

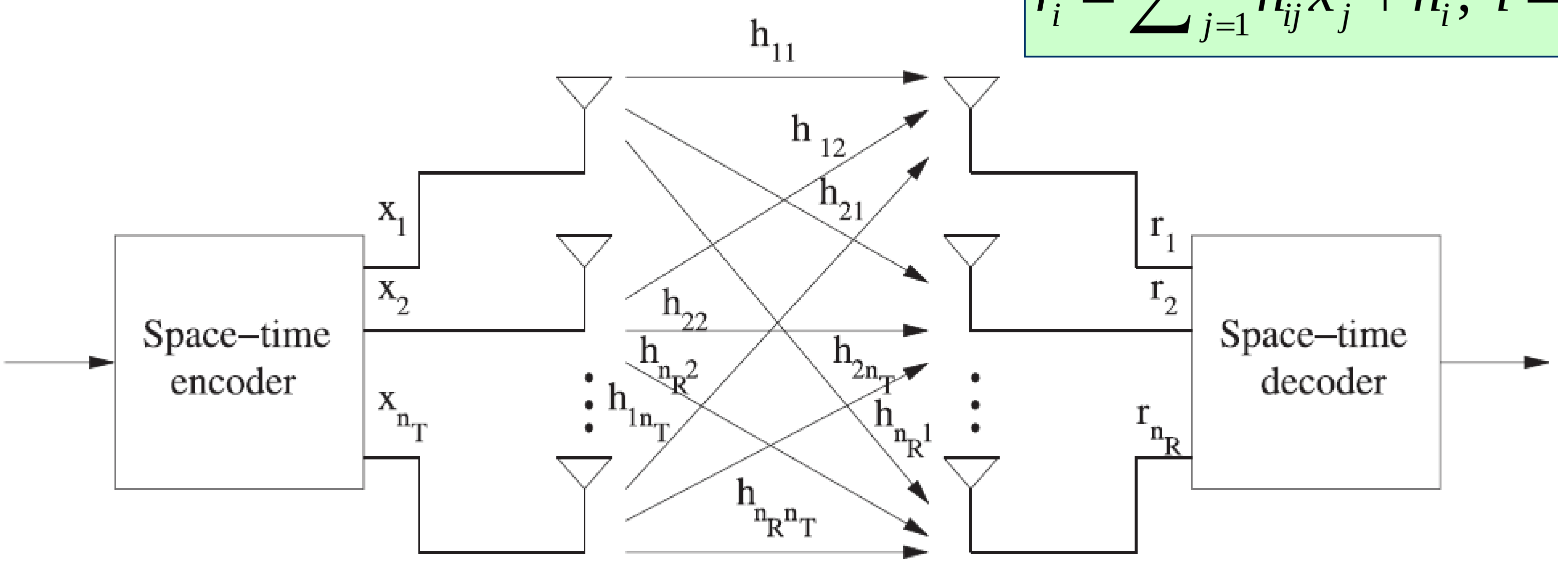
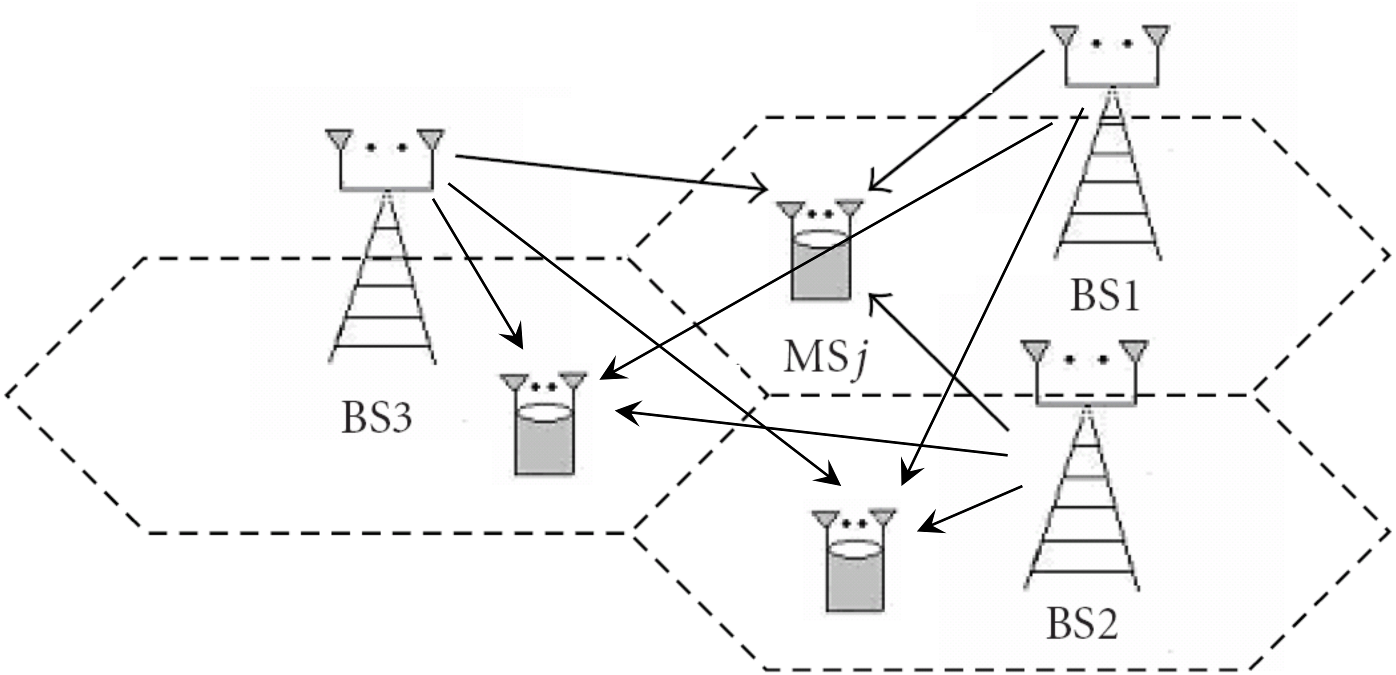


Figure 1.1 Block diagram of a MIMO system

[5] Branka Vuctic, and Jinhong Yuan, "*Space-Time Coding*", Wiley, 2003

2.2.2 MIMO System Model(3/11)

- Multi-cell Multi-user MIMO (or Network MIMO)



[6]Hongyuan Zhang and Huaiyu Dai, "Cochannel interference mitigation and cooperative processing in downlink multicell multiuser MIMO networks," EURASIP Journal on Wireless Communications and Networking, pp.222-235, February 2004.

2.2.2 MIMO System Model(4/11)

- The covariance matrix of the transmitted signal (Recall that x = [x_1, x_2,…, x_{nT}]^T)

$$R_{xx} = E\left\{\mathbf{x}\mathbf{x}^H\right\} = E\begin{bmatrix} x_1x_1^* & x_1x_2^* & \cdots & x_1x_{n_T}^* \\ x_2x_1^* & x_2x_2^* & \cdots & x_2x_{n_T}^* \\ \vdots & \vdots & & \vdots \\ x_{n_T}x_1^* & x_{n_T}x_2^* & \cdots & x_{n_T}x_{n_T}^* \end{bmatrix}$$

The superscript ^{H} denotes the Hermitian operation.

- The total transmitted power is P:

$$P = \mathrm{tr}\left\{R_{xx}\right\} = \sum_{k=1}^{n_T} E\left\{\left|x_k\right|^2\right\}$$

tr(A) denotes the trace of A.

2.2.2 MIMO System Model(5/11)

- If the channel is *unknown* at the transmitter, we assume that the signals transmitted from individual antenna elements have equal powers of P/n_T.
- According to information theory [7], the optimum distribution of transmitted signals is Gaussian.
  - Thus, the elements of x are considered to be "zero mean independently identically distributed (i.i.d.) Gaussian variables".
- Now the covariance matrix of the transmitted signals is given by

$$R_{xx} = \left(P/n_T\right)\mathbf{I}_{n_T}$$

$\mathbf{I}_{nT}$ is the $n_T \times n_T$ identity matrix.

$$r_i = \sum_{j=1}^{n_T} h_{ij} x_j + n_i,\ i = 1, 2, \ldots, n_R$$

2.2.2 MIMO System Model(6/11)

- Assume that the average received power at each of n_R receive branches is equal to the total transmitted power P.
  - Physically, it means that we ignore signal attenuations and amplifications in the propagation process (including path loss, shadowing, antenna gains etc.)
- Thus the normalization constraint for H is:

$$\sum_{j=1}^{n_T} E\left\{\left|h_{ij}\right|^2\right\} = n_T,\ \ i = 1, 2, \ldots, n_R \quad (1.4) \quad \left(\because E\left\{\left|h_{ij}\right|^2\right\} = 1\right)$$

$$\rightarrow E\left\{\left|\sum_{j=1}^{n_T} h_{ij} x_j\right|^2\right\} = \sum_{j=1}^{n_T} E\left\{\left|h_{ij} x_j\right|^2\right\},\ \ i = 1, 2, \ldots, n_R$$

$$= \sum_{j=1}^{n_T} E\left\{\left|h_{ij}\right|^2\right\} E\left\{\left|x_j\right|^2\right\} = n_T\left(P/n_T\right) = P$$

2.2.2 MIMO System Model(7/11)

- The noise at the receiver is described by an $n_R \times 1$ column vector, n.
  - n = $[n_1, n_2, \ldots, n_{nR}]^T$, where the ith component n_i, refers to the received noise at antenna i.
  - $\{n_i\}$ are statistically i.i.d. zero-mean Gaussian variables, with independent and equal variance real and imaginary parts.
- Covariance matrix of noise vector (參考補充教材1-01)

$$R_{nn} = E\left\{\mathbf{nn}^H\right\} = \sigma^2 \mathbf{I}_{n_R}$$

Each of n_R receive branches has identical noise power of σ^2.

2.2.2 MIMO System Model(8/11)

- Matrix-vector form of the received signal vector r $(= [r_1, r_2, \ldots, r_{nR}]^T)$ over n_R receive antennas:

$$\mathbf{r} = \mathbf{Hx} + \mathbf{n}, \text{ where } r_i = \sum_{j=1}^{n_T} h_{ij} x_j + n_i, \ i = 1, 2, \ldots, n_R$$

r_i* : the received signal at antenna *i
***h_{ij}* : the channel fading coefficient from the *j*th transmit antenna to the *i*th receive antenna.**

- The average receive signal-to-noise ratio (SNR) at each receive antenna is defined as:

$$\gamma = E\left[\left|\sum_{j=1}^{n_T} h_{ij} x_j\right|^2\right] \Big/ E\left[\left|n_i\right|^2\right] = \frac{P}{\sigma^2}, \text{ where } E\left[\left|n_i\right|^2\right] = \sigma^2 \text{ for } \forall i$$

2.2.2 MIMO System Model(9/11)

- Example 1.1 2 TX antennas $(n_T = 2)$, 4 RX antennas $(n_R = 4)$

$$\rightarrow \mathbf{H} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{31} & h_{32} \\ h_{41} & h_{42} \end{bmatrix}, \mathbf{x} = [x_1, x_2]^T, \mathbf{n} = [n_1, n_2, n_3, n_4]^T, \mathbf{r} = [r_1, r_2, r_3, r_4]^T$$

$$\rightarrow \mathbf{r} = \mathbf{Hx} + \mathbf{n}, \text{ where } r_i = \sum_{j=1}^{2} h_{ij} x_j + n_i, \ i = 1, 2, \ldots, 4$$

$$\rightarrow \begin{cases} r_1 = h_{11}x_1 + h_{12}x_2 + n_1 \\ r_2 = h_{21}x_1 + h_{22}x_2 + n_2 \\ r_3 = h_{31}x_1 + h_{32}x_2 + n_3 \\ r_4 = h_{41}x_1 + h_{42}x_2 + n_4 \end{cases}$$

Ignoring noise components, there are 2 unknowns in 4 equations.
If H has full rank (full rank = min(n_T,n_R)), there exists a unique solution.
That is, the transmit signal vector $\mathbf{x}$ can be reliably detected if the receive SNR is large enough.

2.2.2 MIMO System Model(10/11)

- Example 1.2 4 TX antennas $(n_T = 4)$, 2 RX antennas $(n_R = 2)$

$$\rightarrow \mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \end{bmatrix}, \mathbf{x} = [x_1, x_2, x_3, x_4]^T, \mathbf{n} = [n_1, n_2]^T, \mathbf{r} = [r_1, r_2]^T$$

$$\rightarrow \mathbf{r} = \mathbf{Hx} + \mathbf{n}, \text{ where } r_i = \sum_{j=1}^{4} h_{ij} x_j + n_i, \; i = 1, 2$$

$$\rightarrow \begin{cases} r_1 = h_{11}x_1 + h_{12}x_2 + h_{13}x_3 + h_{14}x_4 + n_1 \\ r_2 = h_{21}x_1 + h_{22}x_2 + h_{23}x_3 + h_{24}x_4 + n_2 \end{cases}$$

Ignoring noise components, there are 4 unknowns in 2 equations.
There does not exist a unique solution from these 2 equations, even though the channel matrix has full rank.
Thus the same data symbols in signal vector x should be transmitted again in order to obtain two extra equations. (i.e., utilize the temporal domain)
(Note that each of data symbols is resent over different TX antenna from the previous time, if the channel is unchanged. Think why!)

2.2.2 MIMO System Model(11/11)

- Given the channel matrix H, the received signal covariance matrix:

$$
\begin{aligned}
R_{rr} &= E\left\{\mathbf{r}\mathbf{r}^H\right\} = E\left\{\left(\mathbf{H}\mathbf{x}+\mathbf{n}\right)\left(\mathbf{H}\mathbf{x}+\mathbf{n}\right)^H\right\} \\
&= E\left\{\mathbf{H}\mathbf{x}\mathbf{x}^H\mathbf{H}^H+\mathbf{n}\left(\mathbf{H}\mathbf{x}\right)^H+\mathbf{H}\mathbf{x}\mathbf{n}^H+\mathbf{n}\mathbf{n}^H\right\} \\
&= \mathbf{H}R_{xx}\mathbf{H}^H+R_{nn},
\end{aligned}
$$

where the total average received signal power can be expressed as $\mathrm{tr}(R_{rr})$.

2.2.3 MIMO System Capacity Derivation(1/12)

- The system capacity is defined as the "maximum possible transmission rate" such that the "probability of error is arbitrarily small".
- To begin with, assume that the channel matrix H is unknown at the transmitter, but it is only perfectly known at the receiver. (➔CSIR)

2.2.3 MIMO System Capacity Derivation(2/12)

- By the singular value decomposition (SVD) theorem [8], any $n_R \times n_T$ channel matrix H can be written as (參考補充教材1-02)

$$\mathbf{H} = \mathbf{U}\mathbf{D}\mathbf{V}^H$$

The size of D is the same as H.

- $\mathbf{D}$ is an $n_R \times n_T$ non-negative diagonal matrix.

  (Only the first r diagonal elements $\{D_{ii} \neq 0, i = 1, 2, \ldots, r\}$ are nonzero, where r is rank of $\mathbf{H}$, and they are referred to as the nonzero singular values of $\mathbf{H}$.)

- $\mathbf{U}$ and $\mathbf{V}$ are $n_R \times n_R$ and $n_T \times n_T$ unitary matrices, respectively.

$$\rightarrow \mathbf{U}\mathbf{U}^H = \mathbf{I}_{n_R}, \mathbf{V}\mathbf{V}^H = \mathbf{I}_{n_T}$$

2.2.3 MIMO System Capacity Derivation(3/12)

- Note that the diagonal elements of D are the non-negative “square roots of the eigenvalues of matrices HH^H and H^HH”.

$$
\begin{cases}
\mathbf{H}\mathbf{H}^H = \left(\mathbf{U}\mathbf{D}\mathbf{V}^H\right)\left(\mathbf{U}\mathbf{D}\mathbf{V}^H\right)^H = \mathbf{U}\underbrace{\mathbf{D}\mathbf{D}^H}_{n_R \times n_R}\mathbf{U}^H \in C^{n_R \times n_R} \\
\mathbf{H}^H\mathbf{H} = \left(\mathbf{U}\mathbf{D}\mathbf{V}^H\right)^H\left(\mathbf{U}\mathbf{D}\mathbf{V}^H\right) = \mathbf{V}\underbrace{\mathbf{D}^H\mathbf{D}}_{n_T \times n_T}\mathbf{V}^H \in C^{n_T \times n_T}
\end{cases}
$$

$$
\rightarrow
\begin{cases}
\text{The columns of } \mathbf{U} \text{ are the eigenvectors of } \mathbf{H}\mathbf{H}^H. \\
\text{The columns of } \mathbf{V} \text{ are the eigenvectors of } \mathbf{H}^H\mathbf{H}.
\end{cases}
$$

2.2.3 MIMO System Capacity Derivation(4/12)

- The number of nonzero eigenvalues of HH^H and $H^H H$ is equal to the rank of matrix H, denoted by r.
- In addition, both HH^H and $H^H H$ have the same nonzero eigenvalues.
- Let $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ denote the r nonzero eigenvalues of HH^H or $H^H H$.
- Then $\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_r}\}$ are the singular values of H.

2.2.3 MIMO System Capacity Derivation(5/12)

- With SVD, the received vector r can be written as:

$$\mathbf{r} = \mathbf{H}\mathbf{x} + \mathbf{n} = \mathbf{U}\mathbf{D}\mathbf{V}^H \mathbf{x} + \mathbf{n}$$

- Introduce the following transformations:

$$\begin{cases} \mathbf{r}' \equiv [r_1', r_2', \cdots, r_{n_R}']^T = \mathbf{U}^H \mathbf{r} \\ \mathbf{x}' \equiv [x_1', x_2', \cdots, x_{n_T}']^T = \mathbf{V}^H \mathbf{x} \\ \mathbf{n}' \equiv [n_1', n_2', \cdots, n_{n_R}']^T = \mathbf{U}^H \mathbf{n} \end{cases}$$

Since $\mathbf{U}^H$ is invertible, r' is an equivalent version of r.
➔The inverse of $\mathbf{U}^H$ is U.
($\mathbf{U} * \mathbf{U}^H = \mathbf{I}$)
➔ Thus, r = U *r'

The distribution of $\mathbf{n}'$ is the same as $\mathbf{n}$.

2.2.3 MIMO System Capacity Derivation(6/12)

- Transforming the received vector r with the matrix U^H, the equivalent channel matrix is D.

$$\mathbf{r}' = \mathbf{U}^H\mathbf{r} = \mathbf{U}^H\left(\mathbf{UDV}^H\mathbf{x}+\mathbf{n}\right)$$

$$= \mathbf{D}\underbrace{\mathbf{V}^H\mathbf{x}}_{\equiv\mathbf{x}'} + \underbrace{\mathbf{U}^H\mathbf{n}}_{\equiv\mathbf{n}'} = \mathbf{D}\mathbf{x}' + \mathbf{n}'$$

Recall: $\mathbf{r} = \mathbf{H}\mathbf{x}+\mathbf{n}$

- For the $n_R \times n_T$ matrix H, the rank $r \le m = \min(n_R, n_T)$.
- Denote the nonzero singular values of H by

$$\sqrt{\lambda_i}, i = 1, 2, \ldots, r.$$

- The main diagonal vector of the diagonal matrix D is given by

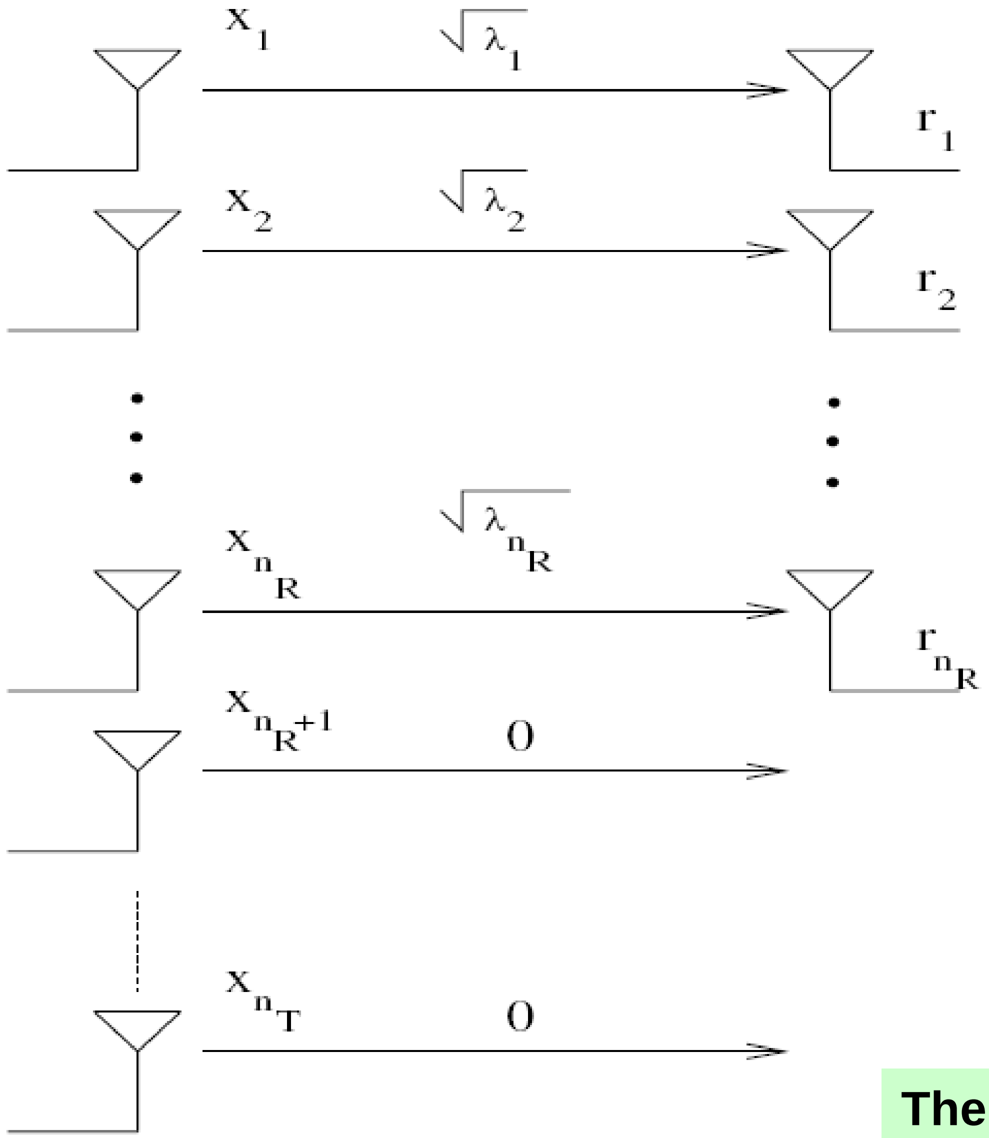
$$\left[\sqrt{\lambda_1}\ \sqrt{\lambda_2}\ \cdots\ \sqrt{\lambda_r}\ 0\ \cdots\ 0\right]^T$$

2.2.3 MIMO System Capacity Derivation(7/12)

- So we can obtain the following *equivalent channels*

$$\begin{cases} r_i' = \sqrt{\lambda_i} x_i' + n_i', & i = 1, 2, \ldots, r \\ r_i' = n_i', & i = r+1, r+2, \ldots, n_R \end{cases} \qquad (1.16)$$

- Thus the original MIMO channel can be considered as r uncoupled parallel sub-channels.
- Each sub-channel is assigned to a singular value $\sqrt{\lambda_i}$ of matrix H, which corresponds to the amplitude channel gain.
- And the channel power gain of a sub-channel corresponds to an eigenvalue λ_i of matrix HH^H.



The rank $r \leq m = \min(n_R, n_T) \leq n_R$

(full row rank)

Figure 1.2 Block diagram of an equivalent MIMO channel if $n_T > n_R$

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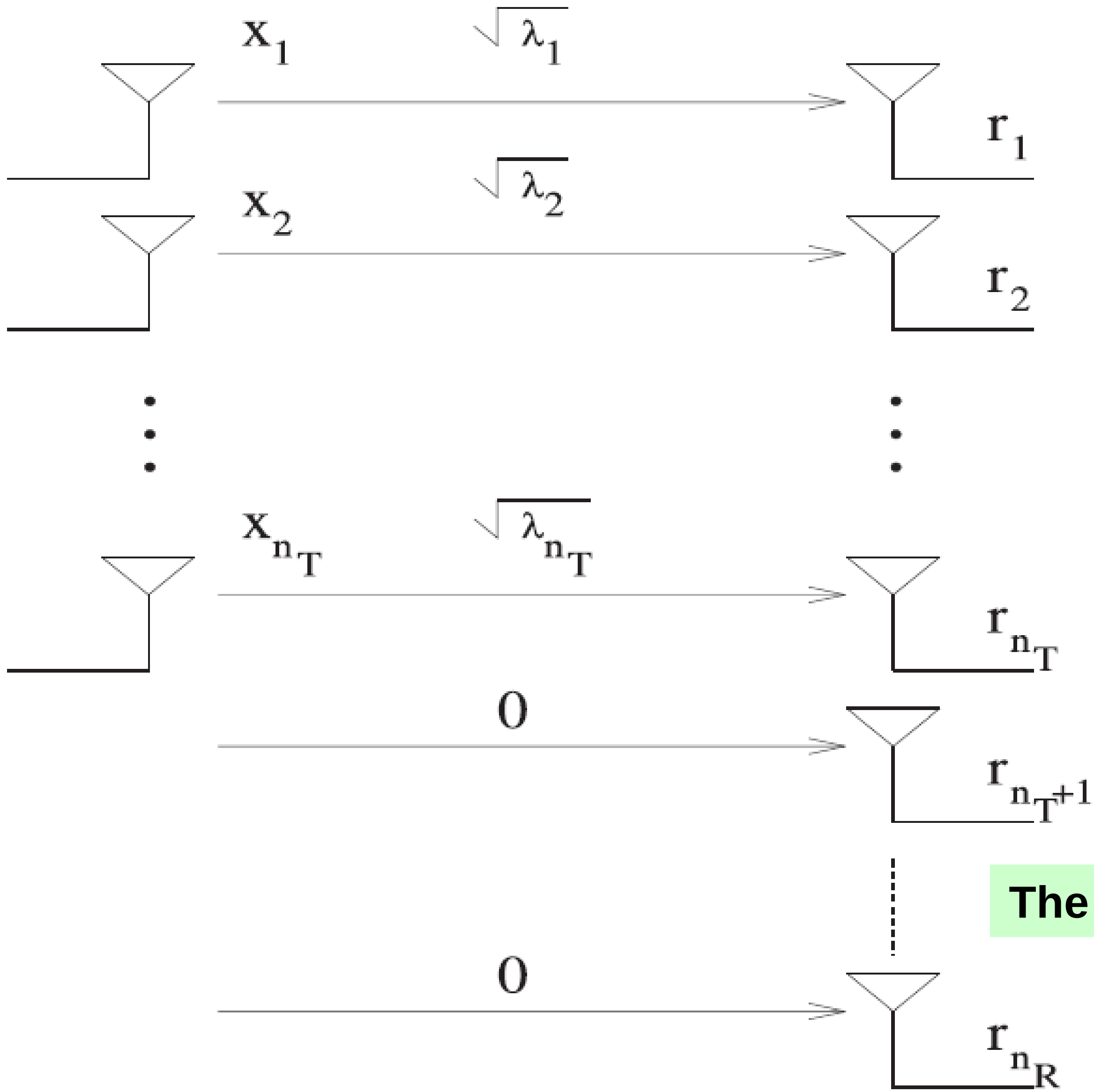
[5] Branka Vuctic, and Jinhong Yuan, "*Space-Time Coding*", Wiley, 2003



The rank $r \le m = \min(n_R, n_T) \le n_T$

Figure 1.3 Block diagram of an equivalent MIMO channel if $n_R > n_T$ **(full column rank)**

National Chung Cheng University

[5] Branka Vuctic, and Jinhong Yuan, "*Space-Time Coding*", Wiley, 2003

2.2.3 MIMO System Capacity Derivation(8/12)

- *Well-conditioned* channel matrix H:
  - $\max\sqrt{\lambda_i} / \min\sqrt{\lambda_i}$ is defined to be the condition number of the channel matrix H.
  - The matrix is said to be well-conditioned if the condition number is close to 1.
  - Well-conditioned channel matrices facilitate communication in the high SNR regime. (See later by Jensen's inequality applied to channel capacity)

2.2.3 MIMO System Capacity Derivation(9/12)

- The covariance matrices and their traces for signals r', x' and n':

$$\mathbf{R}_{r'r'} = \mathbf{U}^H \mathbf{R}_{rr} \mathbf{U}$$

$$\mathbf{R}_{x'x'} = \mathbf{V}^H \mathbf{R}_{xx} \mathbf{V}$$

$$\mathbf{R}_{n'n'} = \mathbf{U}^H \mathbf{R}_{nn} \mathbf{U}$$

$$\left.\begin{aligned} \operatorname{tr}(\mathbf{R}_{r'r'}) &= \operatorname{tr}(\mathbf{R}_{rr}) \\ \operatorname{tr}(\mathbf{R}_{x'x'}) &= \operatorname{tr}(\mathbf{R}_{xx}) \\ \operatorname{tr}(\mathbf{R}_{n'n'}) &= \operatorname{tr}(\mathbf{R}_{nn}) \end{aligned}\right\}$$

Since U and V are unitary matrices, r', x' and n' have the same power as for the original signals, r, x, and n, respectively.
Note that tr(AB) = tr(BA) for square matrices A and B.

2.2.3 MIMO System Capacity Derivation(10/12)

- In the equivalent MIMO channel model (1.16), the sub-channels are uncoupled and thus their capacities add up.

$$\begin{cases} r_i' = \sqrt{\lambda_i}\, x_i' + n_i', & i = 1, 2, \ldots, r \\ r_i' = n_i', & i = r+1, r+2, \ldots, n_R \end{cases} \qquad (1.16)$$

$r \leq m = \min(n_R, n_T)$

- Let W be the bandwidth of each sub-channel and P_{ri} be the received signal power in the ith sub-channel.

$$\rightarrow P_{ri} = \frac{\lambda_i P}{n_T}, \text{ if uniform power allocation (UPA) is applied.}$$

2.2.3 MIMO System Capacity Derivation(11/12)

- By Shannon capacity formula (參考補充教材1-03), the overall MIMO channel capacity with UPA:

$$C \equiv W\sum_{i=1}^{r}\log_2\left(1+SNR_i\right) = W\sum_{i=1}^{r}\log_2\left(1+\frac{P_{ri}}{\sigma^2}\right) \quad (1.19)$$

$$= W\sum_{i=1}^{r}\log_2\left(1+\frac{\lambda_i P}{n_T\sigma^2}\right) \text{ or } = W\log_2\prod_{i=1}^{r}\left(1+\frac{\lambda_i P}{n_T\sigma^2}\right) \text{ bits/sec} \quad (1.21)$$

- By Jensen's inequality (參考補充教材1-04),

$$C = W\sum_{i=1}^{r}\log_2\left(1+\frac{\lambda_i P}{n_T\sigma^2}\right) \le rW\log_2\left\{1+\frac{P}{n_T\sigma^2}\left(\frac{1}{r}\sum_{i=1}^{r}\lambda_i\right)\right\}$$

  - The equality is held, iff the squared singular values $\{\lambda_i, i=1,2,\ldots,r\}$ are all equal.

➔ In this case, the condition number is 1.
➔ The system can achieve maximum capacity, if the channel matrix is well-conditioned.

2.2.3 MIMO System Capacity Derivation(12/12)

- The MIMO capacity formula in (1.21) with UPA can be written as (參考補充教材1-05)

$r \leq m = \min(n_R, n_T)$

$$C = W\log_2 \det\left(\mathbf{I}_m + \frac{P}{n_T\sigma^2}\mathbf{Q}\right), \text{where } \mathbf{Q} = \begin{cases} \mathbf{H}\mathbf{H}^H, & \text{if } m = n_R \\ \mathbf{H}^H\mathbf{H}, & \text{if } m = n_T \end{cases} \quad (1.30)$$

$$= W\log_2\left\{\prod_{i=1}^{m}\left(1 + \frac{\lambda_i P}{n_T\sigma^2}\right)\right\}, \quad \{\lambda_i, i = 1, 2, \ldots, m\} \text{ are the singular values of } \mathbf{Q}$$

Since the nonzero eigenvalues of $\mathbf{HH}^H$ and $\mathbf{H}^H\mathbf{H}$ are the same, the capacities of the channels with matrices $\mathbf{H}$ and $\mathbf{H}^H$ are the same.

- Until now, we derive instantaneous capacity (with fixed channel coefficients).
- If the channel coefficients are random variables, the channel capacity is obtained by averaging over the channel coefficients.

2.2.4 MIMO Channel Capacity Derivation for Adaptive Transmit Power Allocation(1/4)

- When the channel state information (CSI) is known at the transmitter (CSIT), the capacity (1.30) can be increased by adaptively assigning the transmitted power to various antennas according to the "water-filling" rule (or water-pouring rule).
- The rule shows: More power is allocated for the channel in good condition; less for the channel in worse condition.

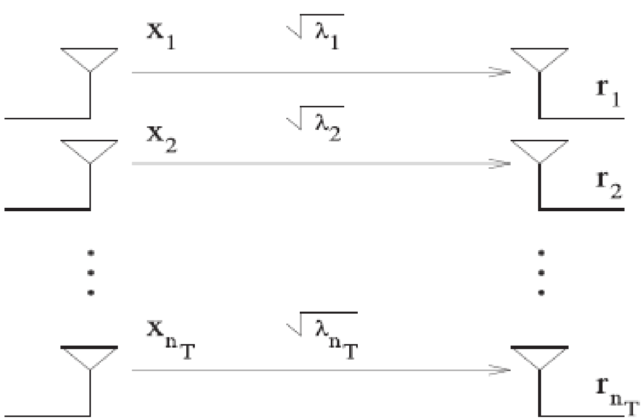
2.2.4 MIMO Channel Capacity Derivation for Adaptive Transmit Power Allocation(2/4)

- The allocated power P_i to transmit antenna i (or sub-channel i) in the equivalent MIMO channel model is given by (參考補充教材1-06):

$$P_i = \left(\mu - \frac{\sigma^2}{\lambda_i} \right)^+ , \quad i = 1, \ldots, r,$$

- a^+ denotes max (a, 0).
- σ^2 denotes average noise power.
- λ_i is the ith squared singular value of H (i.e., channel power gain).

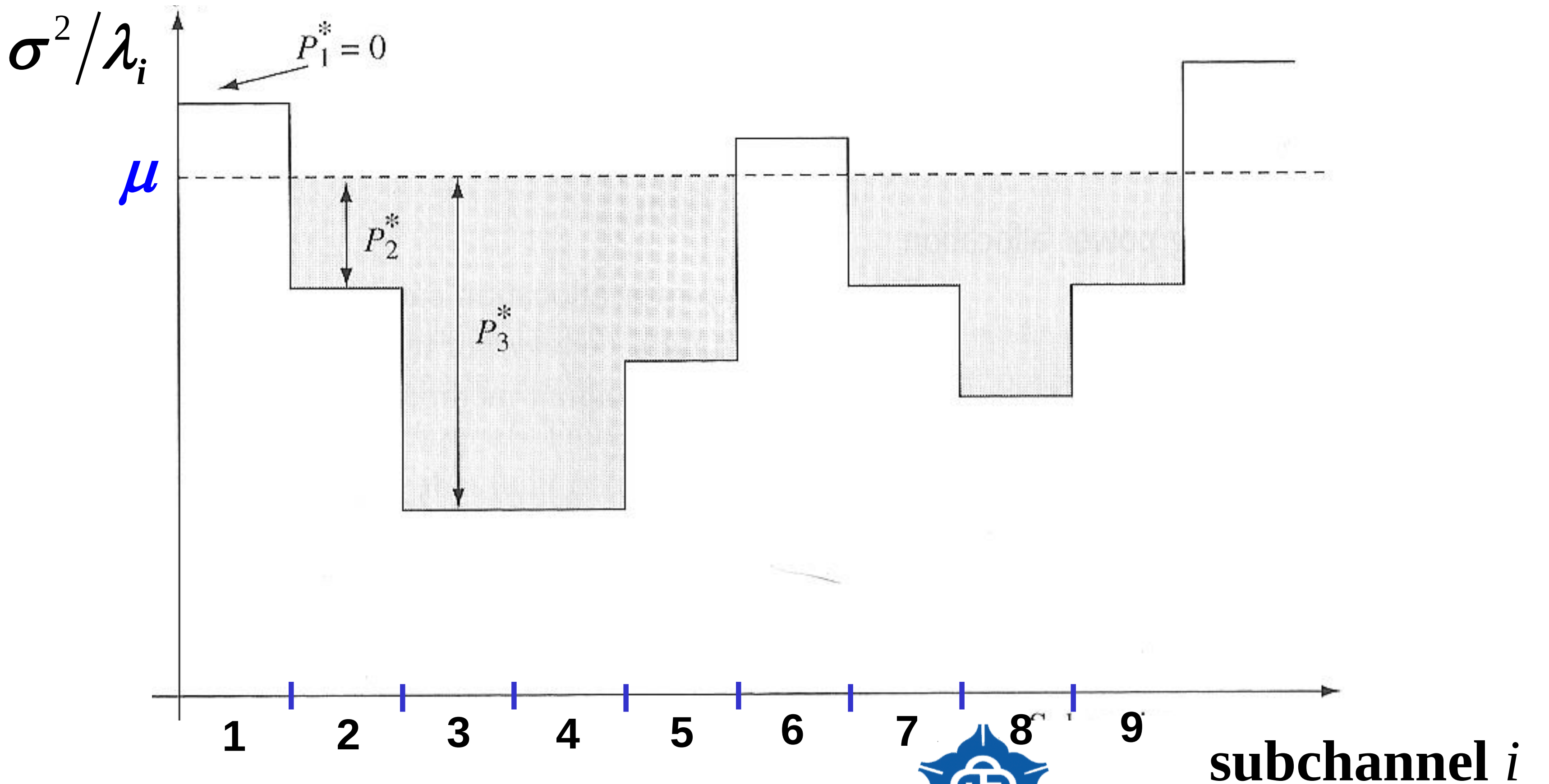
where μ is determined so that $\sum_{i=1}^{r} P_i = P.$



[5] Branka Vuctic, and Jinhong Yuan, “*Space-Time Coding*”, Wiley, 2003

2.2.4 MIMO Channel Capacity Derivation for Adaptive Transmit Power Allocation(3/4)

Illustration of water-filling algorithm



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2.2.4 MIMO Channel Capacity Derivation for Adaptive Transmit Power Allocation(4/4)

- Then, the received signal power at sub-channel *i*:

$$P_{ri} = \lambda_i P_i = \left(\lambda_i \mu - \sigma^2\right)^+, \quad i = 1, 2, \ldots, r$$

- The MIMO channel capacity with adaptive transmit power allocation is

$$P_i = \left(\mu - \frac{\sigma^2}{\lambda_i}\right)^+$$

$$C = W \sum_{i=1}^{r} \log_2 \left(1 + \frac{P_{ri}}{\sigma^2}\right) \qquad (1.19)$$

$$= W \sum_{i=1}^{r} \log_2 \left[1 + \frac{1}{\sigma^2}\left(\lambda_i \mu - \sigma^2\right)^+\right] \qquad (1.35)$$

MIMO的系統容量可由SVD分解後的等效平行通道算出。(補充教材07)

2.2.5 MIMO Capacity Examples for Channels with Fixed Coefficients(1/8)

- Example 2.3: Single Antenna Channel (SISO)

Consider the AWGN channel: $n_T = n_R = 1$ and $H = h = 2$.

$$C = W\log_2\left(1+\frac{P}{\sigma^2}\right) \text{ bits/sec, or } \log_2\left(1+\frac{P}{\sigma^2}\right) \text{ bits/sec/Hz}$$

$$= W\log_2\left(1+SNR_{SISO}\right) \text{ bits/sec,} \qquad (1.30)$$

where $SNR_{SISO} \equiv P/\sigma^2$ denotes received signal-to-noise ratio ($h=1$).

$$\text{If } SNR >> 1 \rightarrow C \approx \log_2(SNR) \text{ bits/sec/Hz}$$

- **A 3 dB increase in *SNR* ($10\log_{10}2*SNR = 3 + 10\log_{10}SNR$) gives an increase of 1 bit/sec/Hz ($\log_2 2*SNR = \log_2 2 + \log_2 SNR = 1 + \log_2 SNR$).**
- **SNR_{dB} = 17 dB (SNR = 50.12) ➔ C = 5.6 bits/sec/Hz**

SNR_{dB} = 20 dB (SNR = 100) ➔ C = 6.6 bits/sec/Hz

2.2.5 MIMO Capacity Examples for Channels with Fixed Coefficients(2/8)

- Example 2.4: A MIMO Channel with Unit Channel Matrix Entries ➔ $h_{ij} = 1,\ i = 1,2,\ldots, n_R;\ j = 1,2,\ldots, n_T$
  1. Repeated Transmission (A signal is transmitted simultaneously over n_T antennas using the same bandwidth.)
  - The received signal part at antenna i:

$$r_i = \sum_{j=1}^{n_T} h_{ij} x = n_T x,\ i = 1, 2, \ldots, n_R,\ \text{ where } E\left\{|x|^2\right\} = P/n_T$$

  - Thus the received signal power at antenna i:

$$P_{r_i} = E\left\{|r_i|^2\right\} = E\left\{|n_T x|^2\right\} = n_T^2 E\left\{|x|^2\right\} = n_T^2 (P/n_T) = n_T P$$

The total transmitted power from n_T transmit antennas is P, but the total received power at one receive antenna is $n_T P$. ➔ The power gain of n_T comes from coherent combining of the transmitted signals.

2.2.5 MIMO Capacity Examples for Channels with Fixed Coefficients(3/8)

Note that the add-up capacity comes from each sub-channel carrying different data stream.

- So, the total received power from n_R receive antennas is
- The

$$P_r = \sum_{i=1}^{n_R} P_{r_i} = n_R n_T P$$

$$\text{Cf.} \quad C = W \sum_{i=1}^{r} \log_2\left(1 + \frac{P_{ri}}{\sigma^2}\right) \ \text{bits/sec} \quad (1.19)$$

$$C = W \log_2\left(1 + n_R n_T \frac{P}{\sigma^2}\right) \ \text{bits/sec}$$

$$SNR_{MIMO} \equiv \frac{n_R n_T P}{\sigma^2} = n_R n_T SNR_{SISO}$$

Cf. SISO:

$$SNR_{SISO} \equiv \frac{P}{\sigma^2}$$

Repeated transmission achieves a diversity gain of $n_T n_R$ relative to a single antenna. Ex., if $n_T = n_R = 8$, and $SNR_{SISO} = 20$ dB, C/W is 12.65 bits/sec/Hz. Cf. $n_T = n_R = 1$, and $SNR_{SISO} = 20$ dB, C/W is 6.658 bits/sec/Hz.

2.2.5 MIMO Capacity Examples for Channels with Fixed Coefficients(4/8)

2. Multiplexing (The signals transmitted at various antennas are different.)
   - The rank of channel matrix H is one in this example. (degree of freedom is only one)
   - So there is only one equivalent channel, and the channel gain (singular value of H) is $\sqrt{(n_T n_R)}$.
   - From (2.30), the received power over the equivalent channel is

$$P_r = \sqrt{n_T n_R}^{\,2}\left(P/n_T\right) = n_R P$$

   - The capacity is

$$C = W\log_2\left(1 + n_R \frac{P}{\sigma^2}\right) \quad \text{bits/sec}$$

Ex., if $n_T = n_R = 8$, and SNR = 20 dB, C/W is 9.646 bits/sec/Hz.
(由於自由度僅有１，故不適合於多工傳送)

Recall: $C = W\log_2 \det\left(\mathbf{I}_m + \frac{P}{n_T\sigma^2}\mathbf{Q}\right) = W\log_2\left\{\prod_{i=1}^{m}\left(1+\frac{\lambda_i P}{n_T\sigma^2}\right)\right\}$ (1.30)

2.2.5 MIMO Capacity Examples for Channels with Fixed Coefficients(5/8)

- Example 2.5: A MIMO Channel with Orthogonal Transmissions
  - If the MIMO channels are orthogonal parallel sub-channels over the same bandwidth, then there is no interference between individual sub-channels.
  - This could be achieved, for example, by spreading transmitted signals from various antenna by orthogonal spreading codes.
  - Considering $n_T = n_R = n$, the channel matrix is

$$\mathbf{H} = \sqrt{n}\mathbf{I}_n$$

The scaling by $\sqrt{n}$ is introduced to satisfy the received power in (2.4).

$$\sum_{j=1}^{n_T} E\left\{\left|h_{ij}\right|^2\right\} = n, \quad i = 1, 2, \ldots, n_R \qquad (1.4)$$

2.2.5 MIMO Capacity Examples for Channels with Fixed Coefficients(6/8)

- Since $HH^H = n\mathrm{I}_n$, using formula (1.30) yields the channel capacity:

$$C = W\log_2 \det\left(\mathbf{I}_n + \frac{P}{n\sigma^2}\mathbf{H}\mathbf{H}^H\right) \quad (1.30)$$

$$= W\log_2 \det\left(\mathbf{I}_n + \frac{P}{n\sigma^2}n\mathbf{I}_n\right)$$

$$= W\log_2\left(1+\frac{P}{\sigma^2}\right)^n = nW\log_2\left(1+\frac{P}{\sigma^2}\right) \text{ bits/sec}$$

Ex., if $n_T = n_R = 8$, and SNR_{SISO} = 20 dB, C/W is 53.264 bits/sec/Hz.

The multiplexing gain n comes from n orthogonal (uncoupled) transmissions using same bandwidth.

2.2.5 MIMO Capacity Examples for Channels with Fixed Coefficients(7/8)

- Example 2.6: Receive Diversity
  - There is single transmit antenna ($n_T = 1$) and n_R receive antennas.
  - The channel matrix is represented by the vector

    $$\mathbf{H} = \left(h_1, h_1, \ldots, h_{nR}\right)^T$$

  - As $n_R > n_T$, formula (1.30) yields the channel capacity

$$C = W \log_2 \det\left(\mathbf{I}_{nT} + \frac{P}{n_T \sigma^2}\mathbf{H}^H \mathbf{H}\right) \quad (1.30)$$

$$= W \log_2\left(1 + \sum_{i=1}^{n_R} \left|h_i\right|^2 \frac{P}{\sigma^2}\right), \quad n_T = 1,\ \mathbf{H}^H\mathbf{H} = \sum_{i=1}^{n_R} \left|h_i\right|^2$$

$$= W \log_2\left(1 + n_R \frac{P}{\sigma^2}\right) \text{ bits/sec,} \quad \text{if } \left|h_i\right|^2 = 1 \text{ for } \forall i$$

This system achieves the diversity gain of n_R relative to a SISO channel.

Ex., if $n_R = 8$, and SNR_{SISO} = 20 dB, C/W is 9.646 bits/sec/Hz.

2.2.5 MIMO Capacity Examples for Channels with Fixed Coefficients(8/8)

- Example 2.7: Transmit Diversity
  - There is single receive antenna (n_R = 1) and n_T transmit antennas.
  - The channel matrix is represented by the vector

    $$\mathbf{H} = \left(h_1, h_1, \ldots, h_{nT}\right)$$

  - As n_T > n_R, (1.30) yields the channel capacity

$$C = W \log_2 \det\left(\mathbf{I}_{n_R} + \frac{P}{n_T \sigma^2}\mathbf{H}\mathbf{H}^H\right), \quad n_R = 1 \quad (1.30)$$

$$= W \log_2\left(1 + \sum_{j=1}^{n_T} \left|h_j\right|^2 \frac{P}{n_T \sigma^2}\right), \quad \mathbf{H}\mathbf{H}^H = \sum\nolimits_{j=1}^{n_T} \left|h_j\right|^2$$

$$= W \log_2\left(1 + \frac{P}{\sigma^2}\right) \text{ bits/sec}, \quad \text{if } \left|h_j\right|^2 = 1 \text{ for } \forall j$$

The capacity does not increase with n_T.
And the capacity is the same as the single antenna channel, because of the power constraint on the n_T transmit antennas.

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