教育部「5G行動寬頻人才培育跨校教學聯盟計畫」
5G行動網路協定與核網技術聯盟中心

課程: 5G系統層模擬技術

第七週：Channel Model(2/2)

Antenna Arrays and Patterns

國立中正大學
National Chung Cheng University

Outline

- Introduction
- Frequency-wavenumber Response and Beam Patterns
- Uniform Linear Arrays
- Array Performance Measures
- Array Steering

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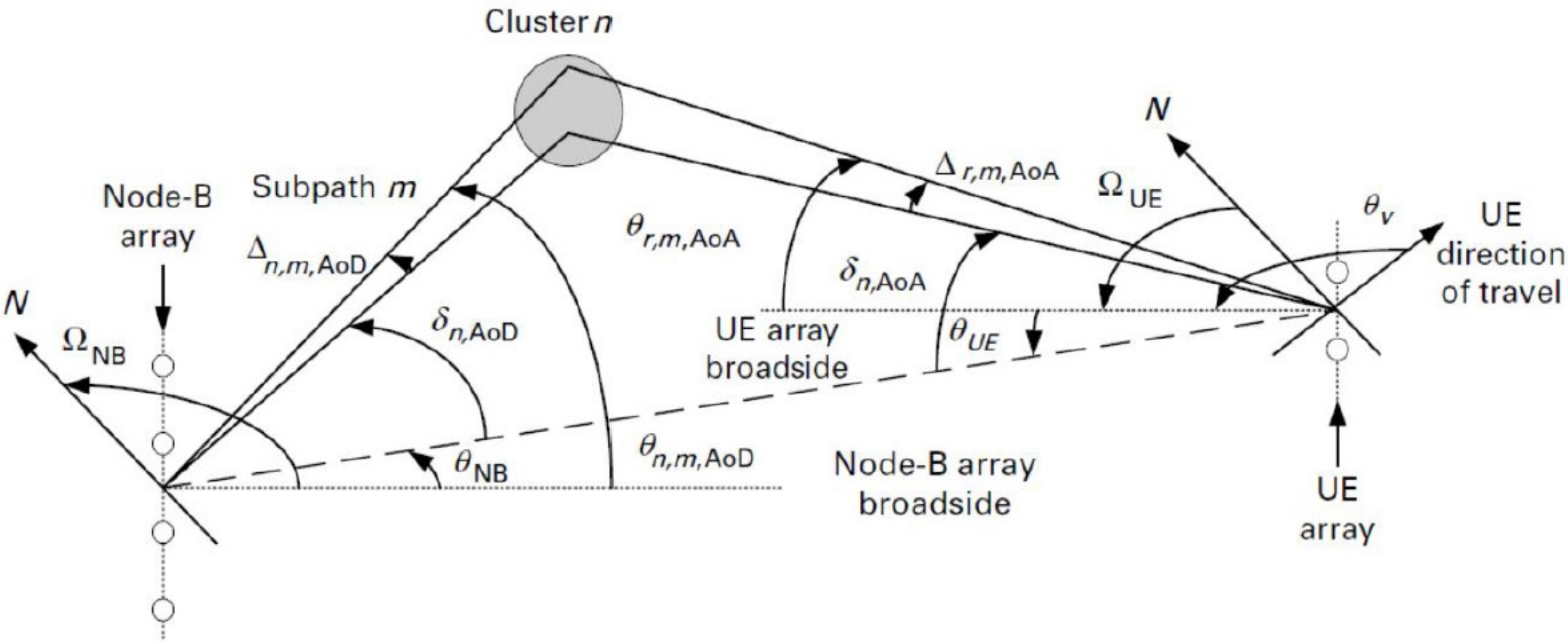
SCM Parameters for Node-B and UE



SCM Parameters(1/2)

- Ω_{NB}, Ω_{UE}: orientations of Node-B and UE between North and broadside.
- $\theta_{n,m,AoD}$: absolute angle of departure (AoD) for the mth sub-path within the nth path in relation to Node-B broadside.
- $\theta_{n,m,AoD} = \theta_{NB}+\delta_{n,AoD}+\Delta_{n,m,AoD}$
- θ_{NB}: LOS AoD direction between eNB and UE relative to the broadside of the eNB array.
- $\delta_{n,AoD}$: AoD for the n_{th} (n = 0, 1, 2, . . . ,N − 1) path relative to the LOS AoD θ_{NB}.
- $\Delta n,m,AoD$:Offset for the m_{th} (m = 0, 1, 2, . . . ,M − 1) sub-path of the nth path relative to $\delta_{n,AoD}$.

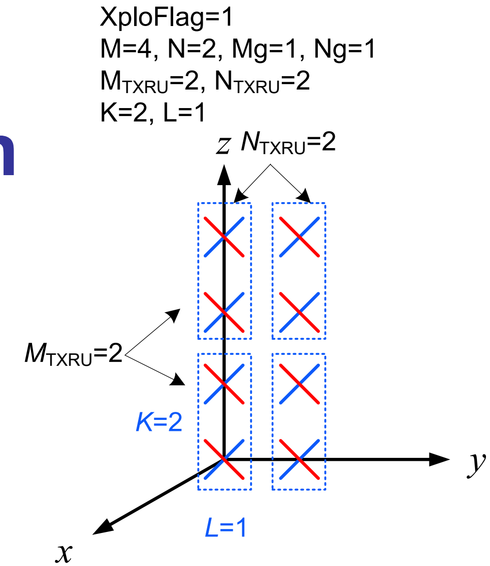
SCM Parameters(2/2)

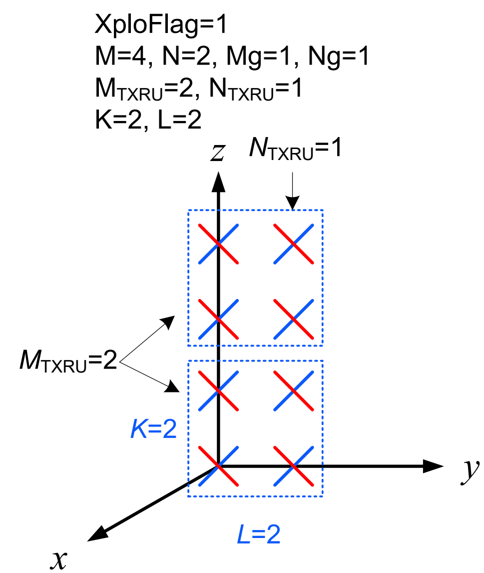
- $\theta_{n,m,AoA}$: absolute angle of arrival (AoA) for the mth sub-path within the nth path in relation to UE broadside.
- $\theta_{n,m,AoA} = \theta_{UE} + \delta_{n,AoA} + \Delta_{n,m,AoA}$
- θ_{UE}: Angle between eNB and UE LOS and the broadside of the UE array.
- $\delta_{n,AoA}$: AoA for the n_{th} (n = 0, 1, 2, . . . ,N − 1) path relative to the LOS.
- $\Delta n,m,AoA$:Offset for the mth (m = 0, 1, 2, . . . ,M − 1) sub-path of the nth path relative to $\delta_{n,AoA}$.
- The angles measured in a clockwise direction are assumed to be negative.
- θ_v: angle of the velocity vector v relative to the UE broadside.

Antenna Configuration

Antenna Configuration：

- M: the number of rows per polarization per panel
- N: the number of columns per polarization per panel
- M_{TXRU}: the number of TXRU per column per polarization per panel
- N_{TXRU}: the number of TXRU per row per polarization per panel
- Polarization flag: 0: V-polarization, 1: X-polarization
- K: the number of virtualization weight per columns for each TXRU
- L: the number of virtualization weight per rows for each TXRU
- d_H: horizontal antenna element spacing
- d_V: vertical antenna element spacing





Introduction

$$\begin{cases} r\text{: radius} \\ \theta\text{: polar angle} \\ \phi\text{: azimuth angle} \end{cases}$$

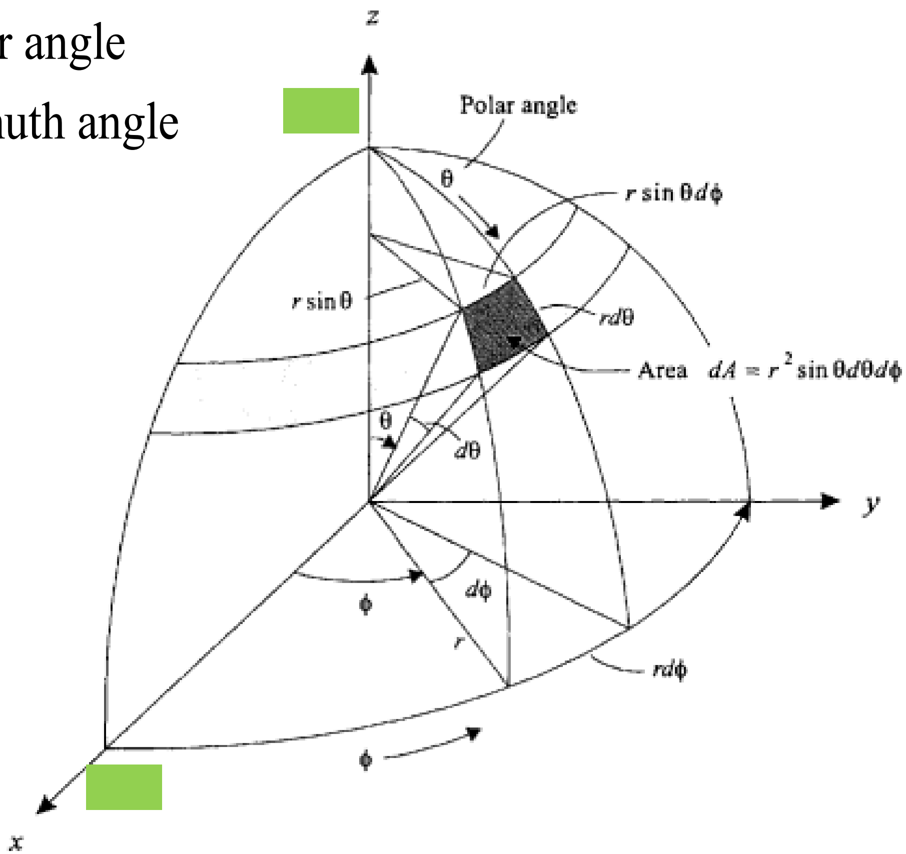


Figure 2.1 Spherical coordinate system.

- The relationship between *Rectangular* and *Spherical* Coordinate System:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

- A unit vector in spherical coordinate system

$$\mathbf{u} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$

[1, p.21]

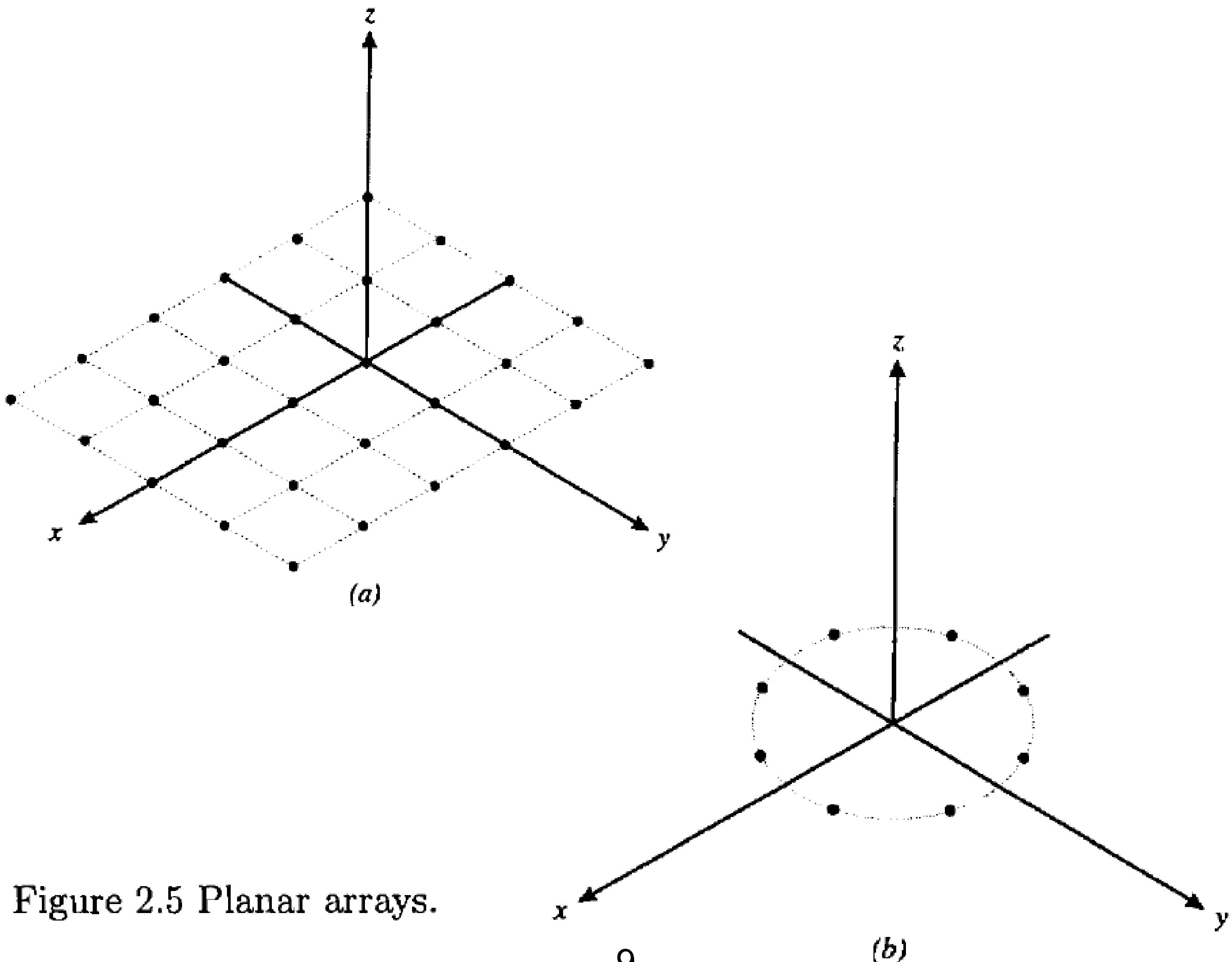


Figure 2.5 Planar arrays.

[1, p.24]

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- Uniform Linear Arrays
- Array Performance Measures
- Array Steering

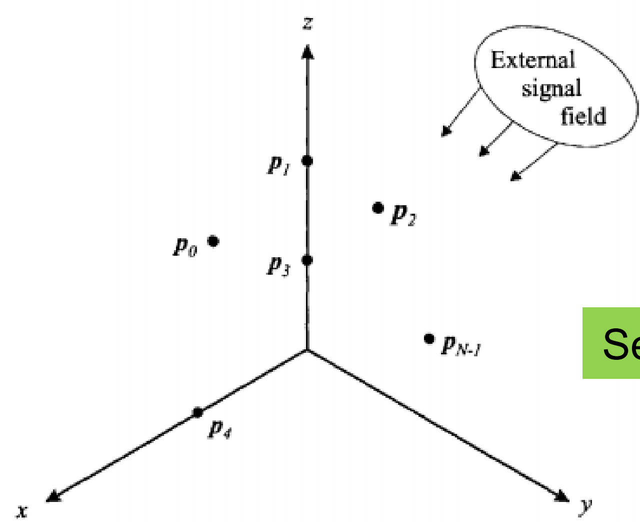
Frequency-wavenumber Response and Beam Patterns(1/17)

- In this section, we analyze the response of an array to an external signal field.
- The sensors spatially sample the signal field at the locations $\mathbf{p}_n, n = 0,1,\ldots,N-1$.
- This yields a vector of received signals denoted by

$$\mathbf{f}(t,\mathbf{p}) = \begin{bmatrix} f(t,\mathbf{p}_0) \\ f(t,\mathbf{p}_1) \\ \vdots \\ f(t,\mathbf{p}_{N-1}) \end{bmatrix}$$

where

$$\mathbf{p} = \{\mathbf{p}_0, \mathbf{p}_1, \cdots$$



Sensor input

[1, p.27]

Figure 2.8 N-Element array.

Frequency-wavenumber Response and Beam Patterns(2/17)

- We process each sensor output by a Linear Time-Invariant(LTI) filter with impulse response $h_n(\tau)$ and sum the outputs to obtain the array output $y(t)$.

$$y(t) = \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} h_n(t-\tau) f(\tau, \mathbf{p}_n) d\tau$$

$$= \int_{-\infty}^{\infty} \mathbf{h}^T(t-\tau) \mathbf{f}(\tau, \mathbf{p}) d\tau$$

where

$$\mathbf{h}(\tau) = \begin{bmatrix} h_0(\tau) \\ h_1(\tau) \\ \vdots \\ h_{N-1}(\tau) \end{bmatrix}$$

[1, p.28]

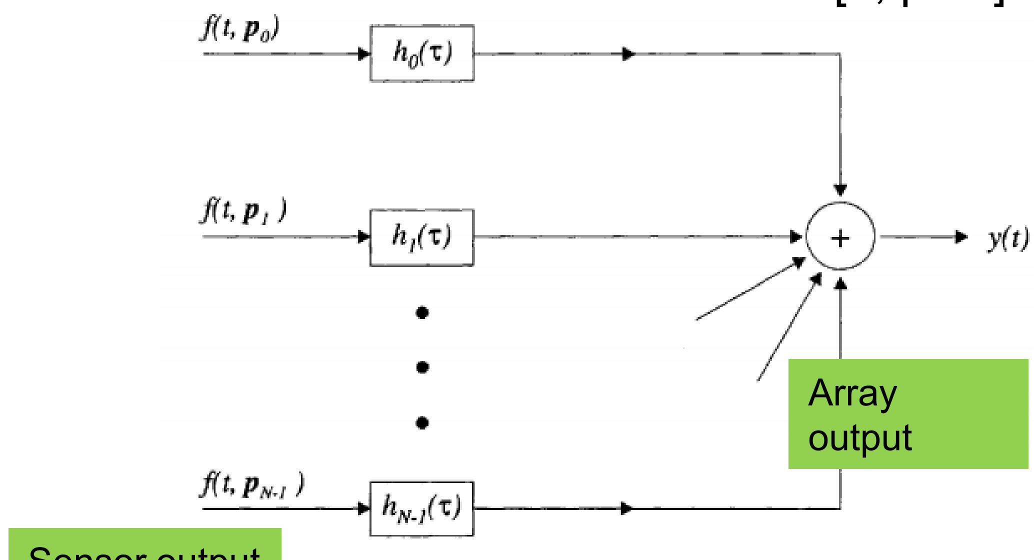


Figure 2.9 Array with linear processing.

Frequency-wavenumber Response and Beam Patterns(3/17)

- The array output in the transform domain is given by

$$\mathbf{Y}(\omega)=\int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt=\mathbf{H}^{T}(\omega)\mathbf{F}(\omega)$$

where

$$\mathbf{H}(\omega)=\int_{-\infty}^{\infty}\mathbf{h}(t)e^{-j\omega t}dt,$$

and

$$\mathbf{F}(\omega,\mathbf{p})=\int_{-\infty}^{\infty}\mathbf{f}(t,\mathbf{p})e^{-j\omega t}dt.$$

In most case, we suppress the **p** dependence on $\mathbf{F}(\omega,\mathbf{p})$ and use $\mathbf{F}(\omega)$.

以上得知，利用Fourier transform，可將sensor收到的signal簡單看出經過LTI filter之後的signal

- $\mathbf{Y}(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$
- $= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h^T(t-\tau) f(\tau, \mathbf{p}) d\tau \, e^{-j\omega t} dt$
- $= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h^T(s) f(\tau, \mathbf{p}) \; e^{-j\omega(s+\tau)} d\tau ds$
- $= \int_{-\infty}^{\infty} h^T(s) e^{-j\omega s} ds \;\; \int_{-\infty}^{\infty} f(\tau, \mathbf{p}) \; e^{-j\omega\tau} d\tau$
- $= \mathbf{H}^T(\omega) \mathbf{F}(\omega, \mathbf{p})$

Let $s = t - \tau$
and $ds = dt$

Frequency-wavenumber Response and Beam Patterns(4/17)

- Consider the input is a plane wave with radian frequency ω propagating in the direction **a** which is a external unit vector,

$$\mathbf{a} = -\mathbf{u} = -[\sin\theta\cos\phi\,, \sin\theta\sin\phi\,, \cos\theta\,]^T,$$

where the minus sign arises because of the direction of **a**.

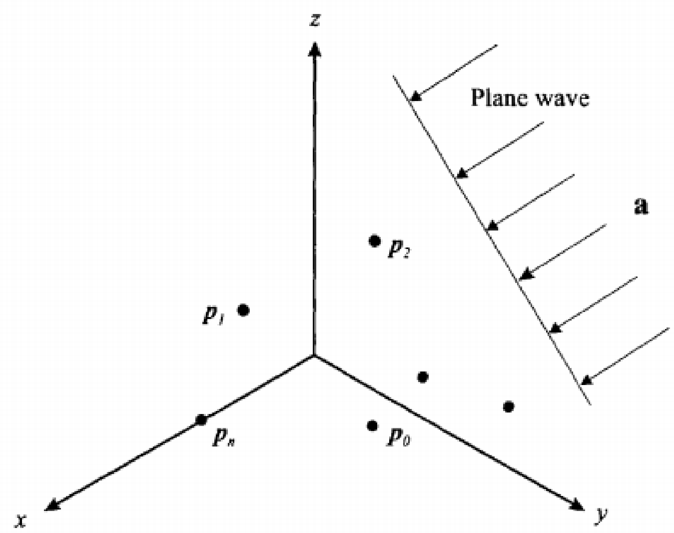


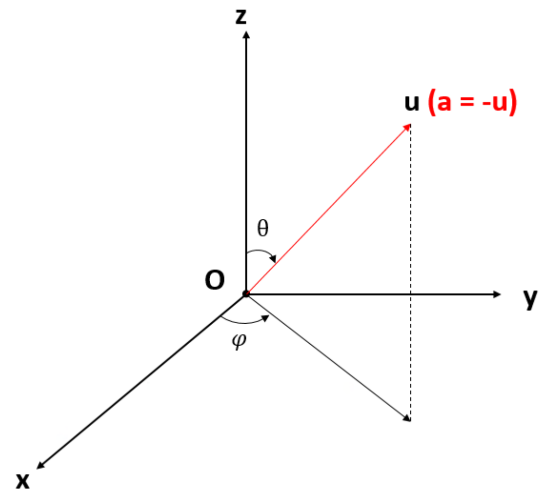
Figure 2.10 Array with plane-wave input.

[1, p.29]

Frequency-wavenumber Response and Beam Patterns(5/17)

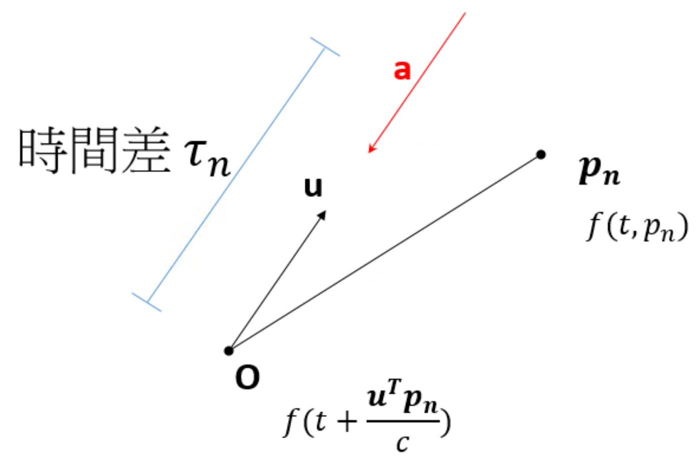
- If $f(t)$ is the signal that would be received at the origin of the coordinate, then the signal received at array is

$$\mathbf{f}(t,\mathbf{p}) \equiv \begin{bmatrix} f(t,\mathbf{p}_0) \\ f(t,\mathbf{p}_1) \\ \vdots \\ f(t,\mathbf{p}_{N-1}) \end{bmatrix} \begin{bmatrix} f(t-\tau_0) \\ f(t-\tau_1) \\ \vdots \\ f(t-\tau_{N-1}) \end{bmatrix}$$



$$\tau_n = \frac{\mathbf{a}^T\mathbf{p}_n}{c} = -\frac{\mathbf{u}^T\mathbf{p}_n}{c}$$

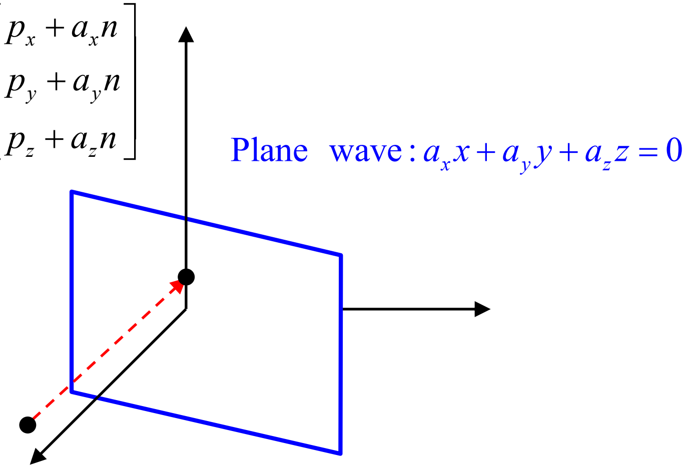
c is the velocity of propagation.



Frequency-wavenumber Response and Beam Patterns(6/17)

- Proof : Delay $\tau = \dfrac{\mathbf{a}^{\mathrm{T}}\mathbf{p}}{c}$

$$\mathbf{p} = \begin{bmatrix} p_x + a_x n \\ p_y + a_y n \\ p_z + a_z n \end{bmatrix}$$



$$\mathbf{p}_n = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

unit normal vector $\mathbf{a} = \begin{bmatrix} -\sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ -\cos\theta \end{bmatrix}$

$$a_x\left(p_x + a_x n\right) + a_y\left(p_y + a_y n\right) + a_z\left(p_z + a_z n\right) = 0$$

$$n = -\frac{a_x p_x + a_y p_y + a_z p_z}{a_x{}^2 + a_y{}^2 + a_z{}^2}$$

$$\left|p_0 - p\right| = \sqrt{\left(a_x n\right)^2 + \left(a_y n\right)^2 + \left(a_z n\right)^2}$$

$$= n\sqrt{\left(a_x\right)^2 + \left(a_y\right)^2 + \left(a_z\right)^2}$$

$$= \frac{a_x p_x + a_y p_y + a_z p_z}{a_x{}^2 + a_y{}^2 + a_z{}^2} \cdot \sqrt{\left(a_x\right)^2 + \left(a_y\right)^2 + \left(a_z\right)^2}$$

$$= \frac{a_x p_x + a_y p_y + a_z p_z}{\sqrt{\left(a_x\right)^2 + \left(a_y\right)^2 + \left(a_z\right)^2}}$$

$$= a_x p_x + a_y p_y + a_z p_z$$

$$= \mathbf{a}^{\mathrm{T}}\mathbf{p}$$

$$\tau_n = \frac{\mathbf{a}^{\mathrm{T}}\mathbf{p}}{c}$$

Frequency-wavenumber Response and Beam Patterns(7/17)

- With the external unit vector **a**, the array output in the transform domain is

$$\mathbf{F}(\omega)=\int_{-\infty}^{\infty}\mathbf{f}(t,\mathbf{p})e^{-i\omega t}dt=\begin{bmatrix}\int_{-\infty}^{\infty}f(t-\tau_0)e^{-j\omega t}dt\\ \int_{-\infty}^{\infty}f(t-\tau_1)e^{-j\omega t}dt\\ \vdots\\ \int_{-\infty}^{\infty}f(t-\tau_{N-1})e^{-j\omega t}dt\end{bmatrix}=\underbrace{\begin{bmatrix}e^{-j\omega\tau_0}\\ e^{-j\omega\tau_1}\\ \vdots\\ e^{-j\omega\tau_{N-1}}\end{bmatrix}}_{\text{Array manifold vector}}\mathbf{F}(\omega)$$

$$\omega\tau_n=\omega\frac{\mathbf{a}^T\mathbf{p}_n}{c}=\omega\frac{-\mathbf{u}^T\mathbf{p}_n}{c}\quad\leftarrow\ \text{Phase shift}$$

$$\mathbf{F}(\omega)\equiv\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$$

Frequency-wavenumber Response and Beam Patterns(8/17)

$\mathbf{f}\left(t,\mathrm{p}_n\right)$的$Fourier\ transform$可寫成

$$\mathbf{F}_n\left(\omega\right)=\int_{-\infty}^{\infty}\mathbf{f}\left(t-\tau_n\right)e^{-j\omega t}dt(\text{令s}=t-\tau_n\text{，ds=dt})$$

$$=\int_{-\infty}^{\infty}\mathbf{f}\left(s\right)e^{-j\omega(\mathrm{s}+\tau_n)}ds$$

$$=e^{-j\omega\tau_n}\int_{-\infty}^{\infty}\mathbf{f}\left(s\right)e^{-j\omega\mathrm{s}}ds$$

$$=e^{-j\omega\tau_n}\mathbf{F}(\omega)\leftarrow\text{在transform後，時間差對信號的影響}$$

Frequency-wavenumber Response and Beam Patterns(9/17)

wavenumber **k**：直覺上把每波長轉換成多少radian，且包含平面波進行方向

- For plane waves, define the wavenumber **k** as

$$\mathbf{k} = \frac{\omega}{c}\mathbf{a} = \frac{2\pi}{\lambda}\mathbf{a} = -\frac{2\pi}{\lambda}\mathbf{u} = -\frac{2\pi}{\lambda}\begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$

λ is the wavelength corresponding to the frequency ω.

$$\omega\tau_n = \mathbf{k}^T\mathbf{p}_n$$

- The magnitude of the wavenumber is $|\mathbf{k}| = 2\pi/\lambda$

- Array manifold vector (steering vector) $\mathbf{v}_{\mathbf{k}}(\mathbf{k})$

$$\mathbf{v}_{\mathbf{k}}(\mathbf{k}) = \begin{bmatrix} e^{-j\omega\tau_0} \\ e^{-j\omega\tau_1} \\ \vdots \\ e^{-j\omega\tau_{N-1}} \end{bmatrix} = \begin{bmatrix} e^{-j\mathbf{k}^T\mathbf{p}_0} \\ e^{-j\mathbf{k}^T\mathbf{p}_1} \\ \vdots \\ e^{-j\mathbf{k}^T\mathbf{p}_{N-1}} \end{bmatrix}$$

The array manifold vector incorporates all of the spatial characteristics of the array.

The subscript k denotes that the argument is in k-space (not array manifold vector).

Frequency-wavenumber Response and Beam Patterns(10/17)

原點O收到的signal的transform

$$\mathbf{F}(\omega)=\begin{bmatrix} e^{-j\omega\tau_0} \\ e^{-j\omega\tau_1} \\ \vdots \\ e^{-j\omega\tau_{N-1}} \end{bmatrix}\boxed{F(\omega)}$$

$$=\mathbf{v}_{\mathbf{k}}(\mathbf{k})F(\omega)$$

利用steering vector可產生antenna array的收到的signal transform

Frequency-wavenumber Response and Beam Patterns(11/17)

- Delay-and-sum beamformer (or conventional beamformer)
  - Shift the input from each sensor so that the signals are aligned in time and add them.
  - In this case, $h_n(\tau) = (1/N)\delta(\tau + \tau_n)$ and $y(t) = f(t)$.

$$\mathbf{h}(\tau) = \frac{1}{N}\begin{bmatrix} \delta(\tau+\tau_0) \\ \delta(\tau+\tau_1) \\ \vdots \\ \delta(\tau+\tau_{N-1}) \end{bmatrix}$$

$$\rightarrow \mathbf{H}(\omega) = \frac{1}{N}\begin{bmatrix} e^{j\omega\tau_0} \\ e^{j\omega\tau_1} \\ \vdots \\ e^{j\omega\tau_{N-1}} \end{bmatrix} = \frac{1}{N}\begin{bmatrix} e^{j\mathbf{k}_s{}^T\mathbf{p}_0} \\ e^{j\mathbf{k}_s{}^T\mathbf{p}_1} \\ \vdots \\ e^{j\mathbf{k}_s{}^T\mathbf{p}_{N-1}} \end{bmatrix} = \frac{1}{N}\mathbf{v}_{\mathbf{k}}^*(\mathbf{k}_s)$$

$\mathbf{k}_s$: the wavenumber of plane-wave signal of interest

[1, p.31]

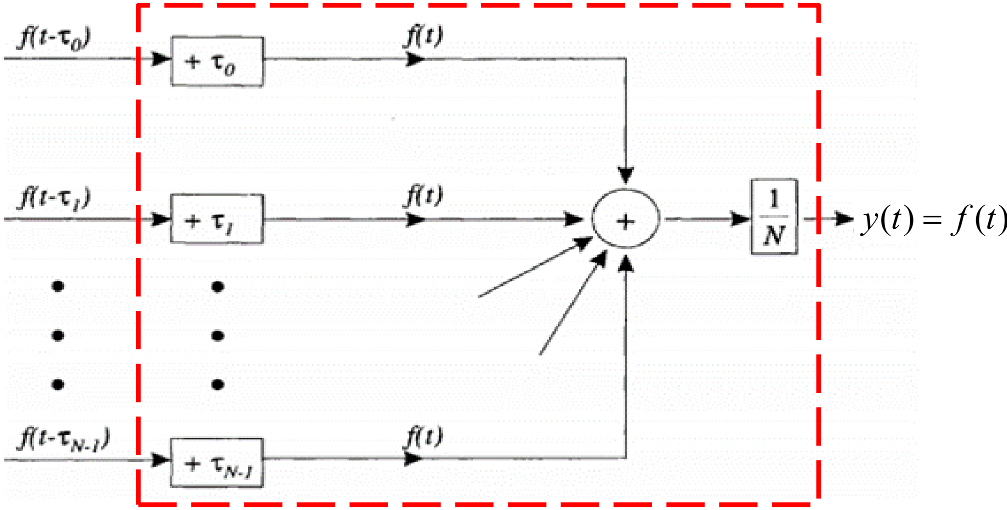


Figure 2.11 Delay-and-sum beamformer.

Frequency-wavenumber Response and Beam Patterns(12/17)

$$h_n(\tau) = \frac{1}{N}\delta(\tau + \tau_n)$$

做$Fourier\ transform$

$$H_n(\omega) = \int_{-\infty}^{\infty} h_n(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} \frac{1}{N}\delta(\tau+\tau_n) e^{-j\omega\tau} d\tau$$

$$= \frac{1}{N} e^{-j\omega(-\tau_n)}$$

$$= \frac{1}{N} e^{jk^T p_n}$$

$$\mathbf{H}^T(\omega) = \frac{1}{N}\mathbf{v}_k^H(\mathbf{k_s})$$

Frequency-wavenumber Response and Beam Patterns(13/17)

- The basis functions of input field are plane waves of the form,

$$f\left(t,\mathbf{p}_n\right)=\exp\left[j\omega\left(t-\tau_n\right)\right]=\exp\left[j\left(\omega t-\mathbf{k}^T\mathbf{p}_n\right)\right],\ n=0,1,\cdots$$

$$\rightarrow \mathbf{f}\left(t,\mathbf{p}\right)=\boxed{\mathbf{v}_{\mathbf{k}}\left(\mathbf{k}\right)}e^{j\omega t}$$

Array manifold vector

- Then the array output becomes

We emphasize the dependence of the output upon the input wavenumber **k** in $y(t,\mathbf{k})$.

$$y\left(t,\mathbf{k}\right)=\int_{-\infty}^{\infty}\mathbf{h}^T\left(\tau\right)\mathbf{f}\left(t-\tau,\mathbf{p}\right)d\tau$$

$$=\underbrace{\mathbf{H}^T\left(\omega\right)\mathbf{v}_{\mathbf{k}}\left(\mathbf{k}\right)}_{\text{complex gain (a scalar)}}e^{j\omega t}=\frac{1}{N}\mathbf{v}_{\mathbf{k}}^{*T}\left(\mathbf{k}_s\right)\mathbf{v}_{\mathbf{k}}\left(\mathbf{k}\right)e^{j\omega t} \qquad (2.35)$$

where $\mathbf{H}\left(\omega\right)$ is the Fourier transform of $\mathbf{h}\left(\tau\right)$.

Frequency-wavenumber Response and Beam Patterns(14/17)

- Accordingly, the array output in the frequency domain can be written as

$$\mathbf{Y}(\omega,\mathbf{k}) = \mathbf{H}^T(\omega)\,\mathbf{v}_{\mathbf{k}}(\mathbf{k})$$

- **Frequency-wavenumber response function** of the array:

$$\Upsilon(\omega,\mathbf{k}) \equiv \mathbf{H}^T(\omega)\,\mathbf{v}_{\mathbf{k}}(\mathbf{k}) \qquad (2.37)$$

  - It describes the *complex gain* of an array to an arbitrary input plane wave with wavenumber **k** and temporal frequency ω.

- **Beam pattern**: $$B(\omega:\theta,\phi) = \Upsilon(\omega,\mathbf{k})\Big|_{\mathbf{k}=\frac{2\pi}{\lambda}\mathbf{a}(\theta,\phi)}$$

  - The beam pattern is the frequency-wavenumber response function evaluated versus the direction.
  - $\mathbf{a}(\theta,\phi)$ is a unit vector with spherical coordinate angles $\{\theta,\phi\}$.

Frequency-wavenumber Response and Beam Patterns(15/17)

- Let ΔT_{max} be the maximum travel time between any two elements in the array; B_s be the bandwidth of input plane wave.
- If input signals satisfy $B_s \cdot \Delta T_{max} \ll 1$, then they are referred to as *narrowband signals*.
- In the narrowband case, the delay is approximated by a phase shift. This implementation is commonly referred to as a **phased array** and is widely used in practice.

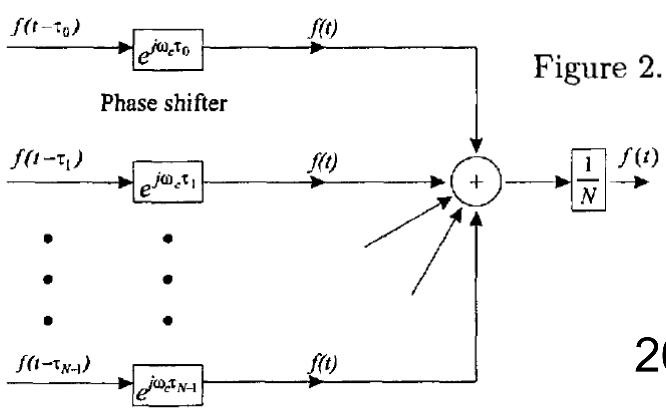


Figure 2.12 Narrowband beamformer implemented using phase shifters.

[1, p.35]

Frequency-wavenumber Response and Beam Patterns(16/17)

- In many applications, we want to adjust the gain and phase of the output of each sensor to achieve a desirable beam pattern.
- Define the complex weight vector as follows.

$$\mathbf{w} = \begin{bmatrix} w_0 & w_1 & \cdots & \end{bmatrix}^T$$

A complex weight can represent the gain and phase.

- Accordingly,

$$\mathbf{w}^H = \mathbf{H}^T(\omega)$$

$$(2.35) \quad \rightarrow \quad \mathbf{y}(t,\mathbf{k}) = \mathbf{H}^T(\omega)\mathbf{v_k}(\mathbf{k})e^{j\omega t} = \mathbf{w}^H\mathbf{v_k}(\mathbf{k})e^{j\omega t} \; (\text{當f(t,p)} = \mathbf{v_k}(\mathbf{k})e^{j\omega t})$$

$$(2.37) \quad \rightarrow \quad \Upsilon(\omega,\mathbf{k}) = \mathbf{w}^H\mathbf{v_k}(\mathbf{k})$$

Frequency-wavenumber Response and Beam Patterns(17/17)

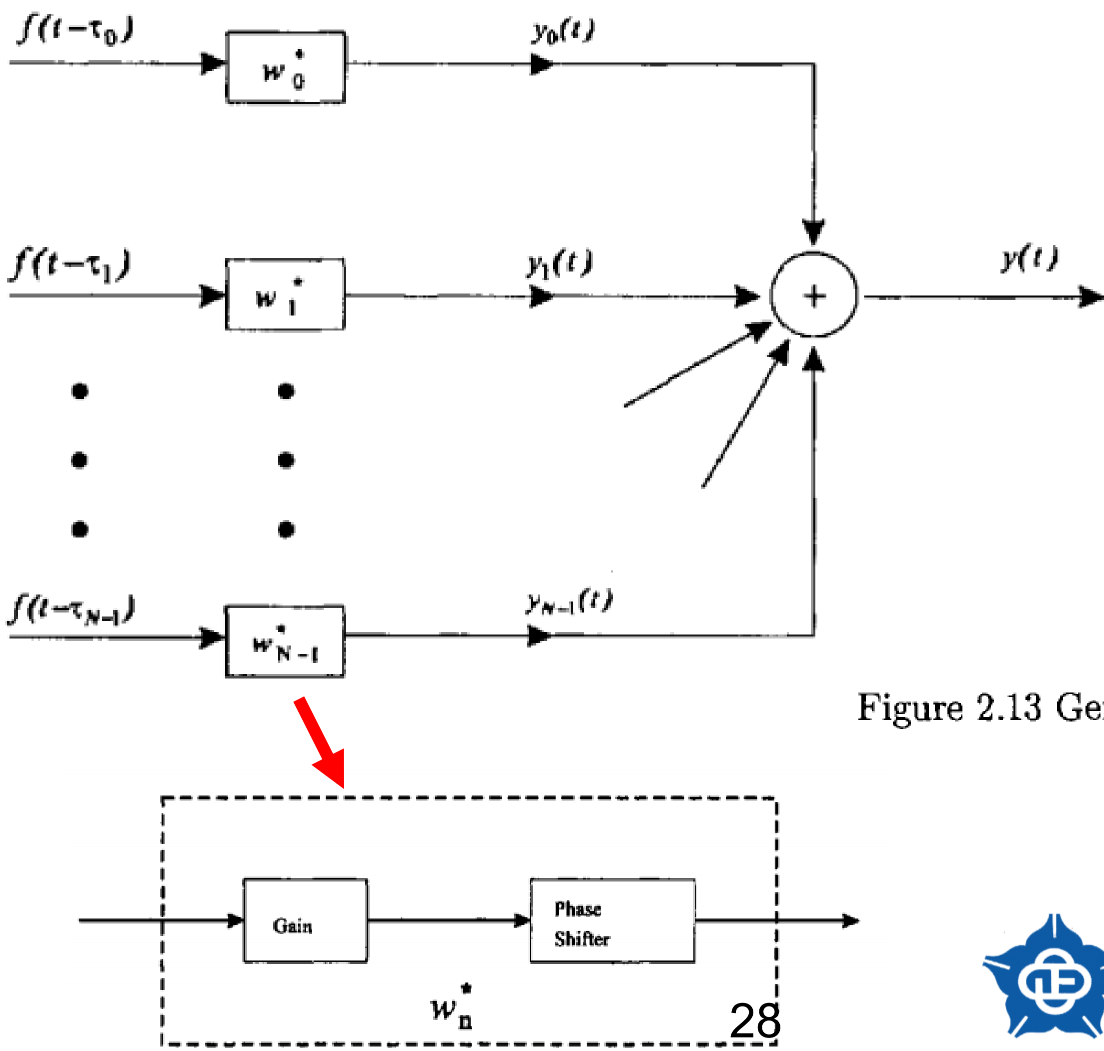


Figure 2.13 General narrowband beamformer.

[1, p.36]

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$$\mathbf{v}_{\mathbf{k}}(\mathbf{k}) = \begin{bmatrix} e^{-j\mathbf{k}^T\mathbf{p}_0} \\ e^{-j\mathbf{k}^T\mathbf{p}_1} \\ \vdots \\ e^{-j\mathbf{k}^T\mathbf{p}_{N-1}} \end{bmatrix} \quad \mathbf{k} = -\frac{2\pi}{\lambda}\begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$

Uniform Linear Arrays(1/17)

- Uniform linear array (ULA)
  - There are *N* elements located on the *z*-axis with uniform spacing equal to *d*.
  - We have placed the center of the array at the origin of the coordinate system for computational advantages.
  - The locations of the elements are

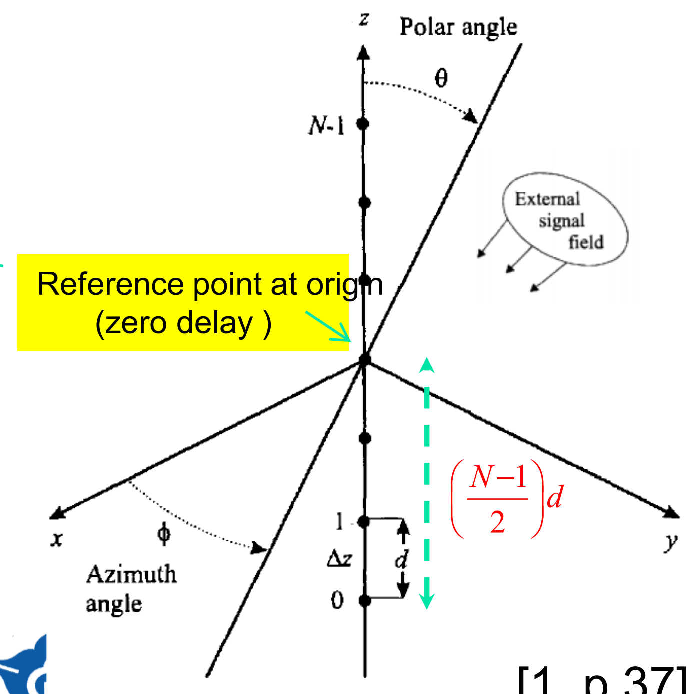
$$\begin{cases} p_{z_n} = \left(n - \dfrac{N-1}{2}\right)d, \quad n = 0,1,\cdots \\ p_{x_n} = p_{y_n} = 0. \end{cases}$$

  - Then the array manifold vector is

$$\mathbf{v}_{\mathbf{k}}(k_z) = \begin{bmatrix} e^{j\left(\frac{N-1}{2}\right)k_z d} & e^{j\left(\frac{N-1}{2}-1\right)k_z d} & \cdots & e^{-j\left(\frac{N-1}{2}\right)k_z d} \end{bmatrix}^T$$

where $k_z = -\dfrac{2\pi}{\lambda}\cos\theta$

Note that the linear array has no resolution capability in the ϕ-direction.



[1, p.37]

Figure 2.14 Linear array along z-axis.

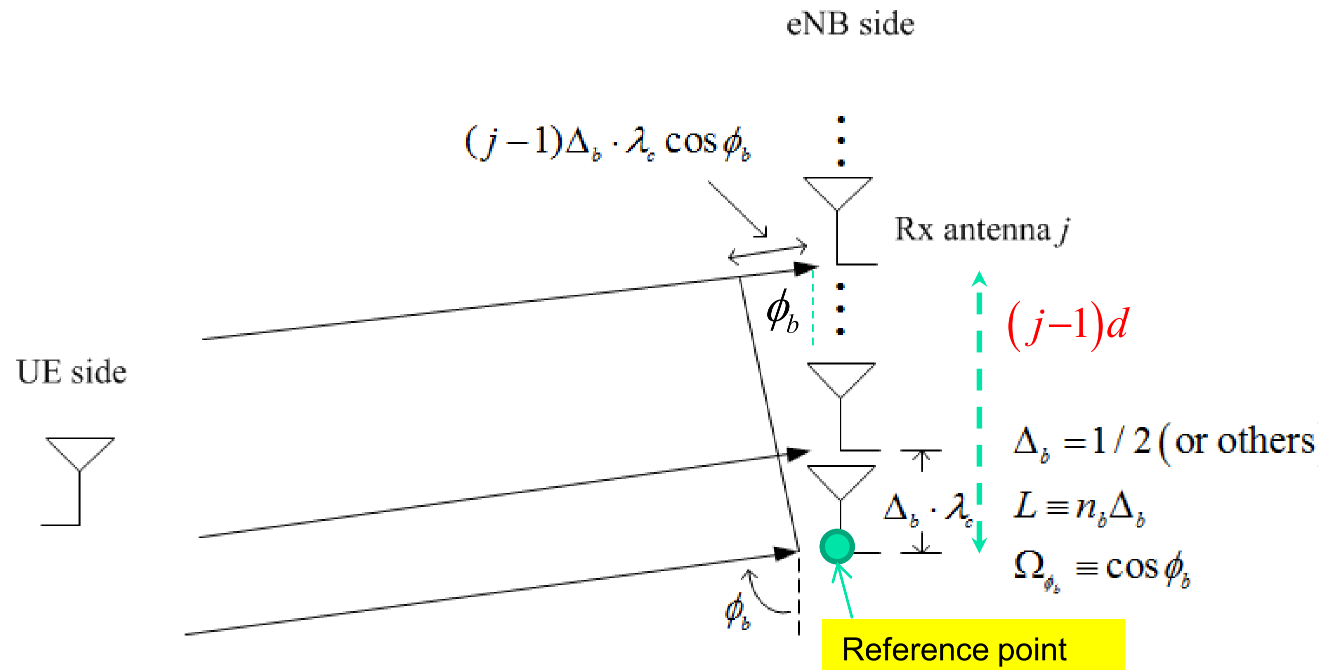
Uniform Linear Arrays(2/17)

- Example

$\mathbf{v}_{\mathbf{k}}(\mathbf{k})$第j項

$$-\omega\tau_j = -2\pi f\frac{(j-1)\Delta_b\lambda\cos\phi_b}{c} = -2\pi f\frac{(j-1)d\cos\phi_b}{f\lambda} = -2\pi\frac{(j-1)d\cos\phi_b}{\lambda}$$

where $d = \Delta_b\lambda$



n_b : number of antennas at eNB

ULAs cannot distinguish the regions between $0 \le \theta \le \pi$ and $\pi \le \theta \le 2\pi$.

Uniform Linear Arrays(3/17)

- Frequency-wavenumber response function of ULA:

$$\Upsilon(\omega, k_z) = \mathbf{w}^H \mathbf{v}_{\mathbf{k}}(k_z) = \sum_{n=0}^{N-1} w_n^* e^{-j\left(n-\frac{N-1}{2}\right)k_z d} = \sum_{n=0}^{N-1} w_n^* e^{j\left(n-\frac{N-1}{2}\right)\frac{2\pi d}{\lambda}\cos\theta} \quad (2.58)$$

$$(2.37): \Upsilon(\omega, \mathbf{k}) \equiv \mathbf{H}^T(\omega)\mathbf{v}_{\mathbf{k}}(\mathbf{k})$$

$$\mathbf{v}_{\mathbf{k}}(\mathbf{k}) = \begin{bmatrix} e^{-j\omega\tau_0} \\ e^{-j\omega\tau_1} \\ \vdots \\ e^{-j\omega\tau_{N-1}} \end{bmatrix} = \begin{bmatrix} e^{-j\mathbf{k}^T\mathbf{p}_0} \\ e^{-j\mathbf{k}^T\mathbf{p}_1} \\ \vdots \\ e^{-j\mathbf{k}^T\mathbf{p}_{N-1}} \end{bmatrix}, \quad \mathbf{k} = -\frac{2\pi}{\lambda}\begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$

令$\mathrm{p}_{n,x}$、$\mathrm{p}_{n,\mathrm{y}} = 0$、$\mathrm{p}_{n,\mathrm{z}} = (n - \frac{N-1}{2})d$

$$\mathbf{v}_{\mathbf{k}}(\mathbf{k}) = \begin{bmatrix} e^{-j\mathbf{k}_z(0-\frac{N-1}{2})d} \\ e^{-j\mathbf{k}_z(1-\frac{N-1}{2})d} \\ \vdots \\ e^{-j\mathbf{k}_z(\mathrm{N}-1-\frac{N-1}{2})d} \end{bmatrix} \text{ and } \mathrm{k}_z = -\frac{2\pi}{\lambda}\cos\theta$$

Uniform Linear Arrays(4/17)

- Define

$$\psi = -k_z d = \frac{2\pi}{\lambda}\cos\theta \cdot d = \frac{2\pi d}{\lambda} u_z$$

準備要來畫beam pattern

where $u_z = \cos\theta$ is the directional cosine w.r.t. the z-axis.

- Then the frequency-wavenumber function in ψ-space:

$$\Upsilon_\psi(\psi) = e^{-j\frac{N-1}{2}\psi} \sum_{n=0}^{N-1} w_n^* e^{jn\psi}, \quad -\frac{2\pi d}{\lambda} \le \psi \le \frac{2\pi d}{\lambda}$$

  - The propagation signals are in the **visible region** where $0 \le \theta \le \pi$, which implies $-2\pi d/\lambda \le \psi \le 2\pi d/\lambda$.

Uniform Linear Arrays(5/17)

- Array manifold vector in θ, u and ψ space:

$$\begin{cases} \left[\mathbf{v}_\theta(\theta)\right]_n = e^{j\left(n-\frac{N-1}{2}\right)\frac{2\pi d}{\lambda}\cos\theta} \\ \left[\mathbf{v}_u(u)\right]_n = e^{j\left(n-\frac{N-1}{2}\right)\frac{2\pi d}{\lambda}u} \\ \left[\mathbf{v}_\psi(\psi)\right]_n = e^{j\left(n-\frac{N-1}{2}\right)\psi} \end{cases}, \quad n = 0, 1, \cdots$$

- For ULAs, we normally write the array manifold vector in terms of ψ.

Uniform Linear Arrays(6/17)

- The beam pattern in three different forms:

$$B_\theta(\theta) = \mathbf{w}^H \mathbf{v}_\theta(\theta) = \sum_{n=0}^{N-1} w_n^* e^{j\left(n-\frac{N-1}{2}\right)\frac{2\pi d}{\lambda}\cos\theta}, \quad 0 \le \theta \le \pi$$

$$B_u(u) = \mathbf{w}^H \mathbf{v}_u(u) = \sum_{n=0}^{N-1} w_n^* e^{j\left(n-\frac{N-1}{2}\right)\frac{2\pi d}{\lambda}u}, \quad -1 \le u \le 1$$

$$B_\psi(\psi) = \mathbf{w}^H \mathbf{v}_\psi(\psi) = \sum_{n=0}^{N-1} w_n^* e^{j\left(n-\frac{N-1}{2}\right)\psi}, \quad -\frac{2\pi d}{\lambda} \le \psi \le \frac{2\pi d}{\lambda}$$

Uniform Linear Arrays(7/17)

- Example (Beam pattern in θ space)
  - If the signal arrives from a signal direction $\tilde{\theta}$, then the optimal receiver projects the received signal onto the vector

$$\mathbf{w}^H(\tilde{\theta}) = \frac{1}{N}\mathbf{v}_\theta^H(\tilde{\theta}),$$

where $[\mathbf{v}_\theta(\theta)]_n = e^{j\left(n-\frac{N-1}{2}\right)\frac{2\pi d}{\lambda}\cos\theta}, n = 0,1, \ldots N-1, \quad \theta \in (0, \pi]$.

  - A signal from any other direction ($\theta \neq \tilde{\theta}$) is attenuated by a factor of $B_\theta\left(\tilde{\theta}\right) = \mathbf{w}^H(\tilde{\theta})\mathbf{v}_\theta(\theta)$.

Uniform Linear Arrays(8/17)

- Example (Beam pattern in θ space) (Cont.)
  - Derivation of beam pattern

$$B_\theta(\tilde{\theta}) \cdots \sum_{n=0}^{N-1}\frac{1}{N} e^{-j\left(n-\frac{N-1}{2}\right)\frac{2\pi d}{\lambda}\cos\tilde{\theta}} e^{j\left(n-\frac{N-1}{2}\right)\frac{2\pi d}{\lambda}\cos\theta}$$

$$=\frac{1}{N}\sum_{n=0}^{N-1} e^{j\left(n-\frac{N-1}{2}\right)\frac{2\pi d}{\lambda}(\cos\theta-\cos\tilde{\theta})} = \frac{1}{N} e^{-j\frac{N-1}{2}\times\frac{2\pi d}{\lambda}\Omega}\sum_{n=0}^{N-1} e^{jn\frac{2\pi d}{\lambda}\Omega} \text{(等比級數}\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}\text{)}$$

$$=\frac{1}{N} e^{-j\frac{(N-1)\pi d\Omega}{\lambda}}\left(\frac{1\text{-}e^{jN\frac{2\pi d}{\lambda}\Omega}}{1\text{-}e^{j\frac{2\pi d}{\lambda}\Omega}}\right)$$

where $\Omega_\theta \equiv \cos\theta$ and $\Omega \equiv \cos\theta - \cos\tilde{\theta}$ … $\Omega_{\tilde{\theta}}$

$$=\frac{1}{N}\frac{e^{-jN\frac{\pi d}{\lambda}\Omega}}{e^{-j\frac{\pi d}{\lambda}\Omega}}\frac{1-e^{jN\frac{2\pi d}{\lambda}\Omega}}{1-e^{j\frac{2\pi d}{\lambda}\Omega}} = \frac{1}{N}\frac{e^{-jN\frac{\pi d}{\lambda}\Omega}-e^{jN\frac{\pi d}{\lambda}\Omega}}{e^{-j\frac{\pi d}{\lambda}\Omega}-e^{j\frac{\pi d}{\lambda}\Omega}}$$

$$=\frac{1}{N}\sin\left(N\frac{\pi d}{\lambda}\Omega\right)\Big/\sin\left(\frac{\pi d}{\lambda}\Omega\right)$$

N is the number of antenna elements.

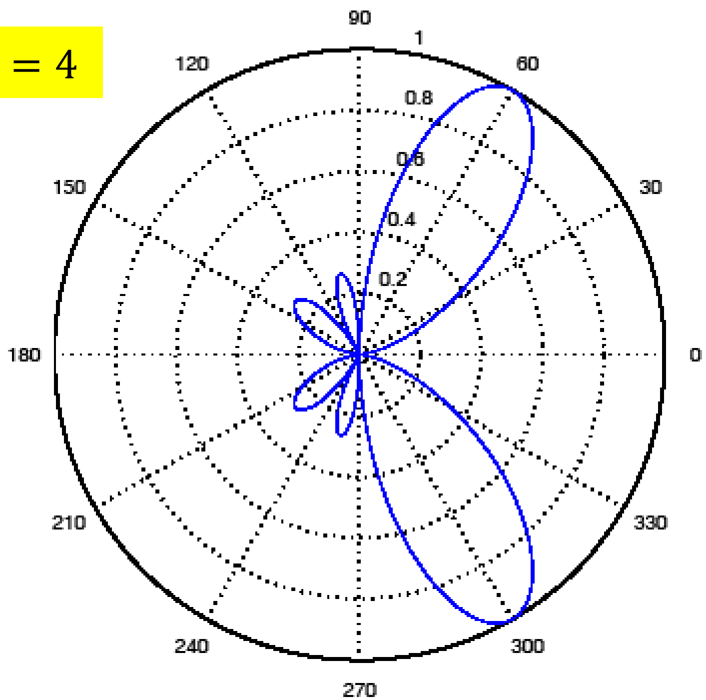
Uniform Linear Arrays(9/17)

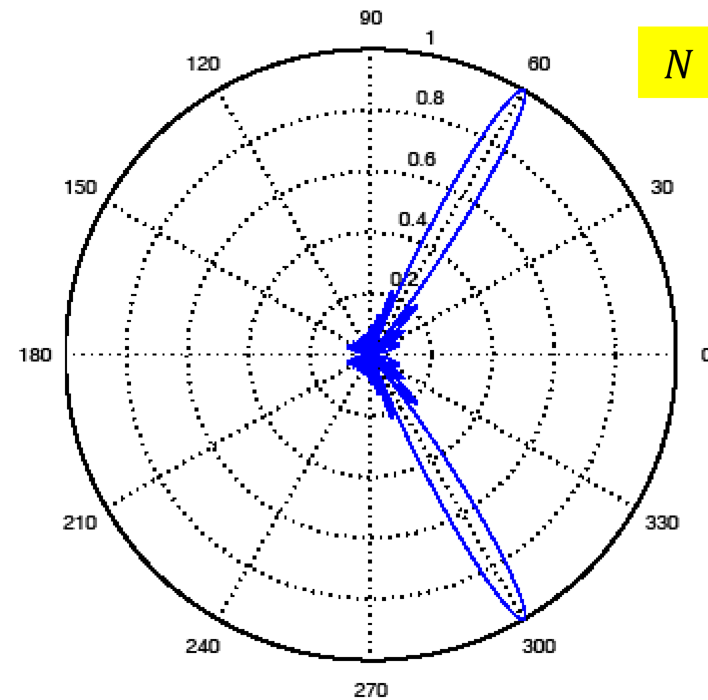
- Example (Beam pattern in θspace) (Cont.)
  - Beam pattern is the plot

$$\left(\theta, \left|B_\theta\left(\tilde{\theta} \quad \right), 2\pi\right)$$

Signal direction $\tilde{\theta} = 60^o$; Inter-element spacing $d = \lambda/2$

$N = 4$



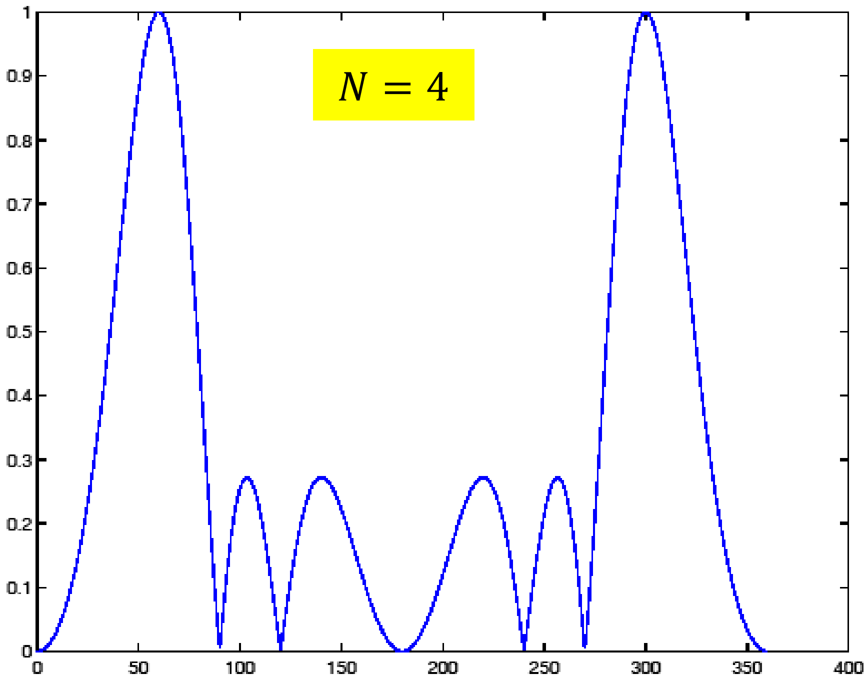


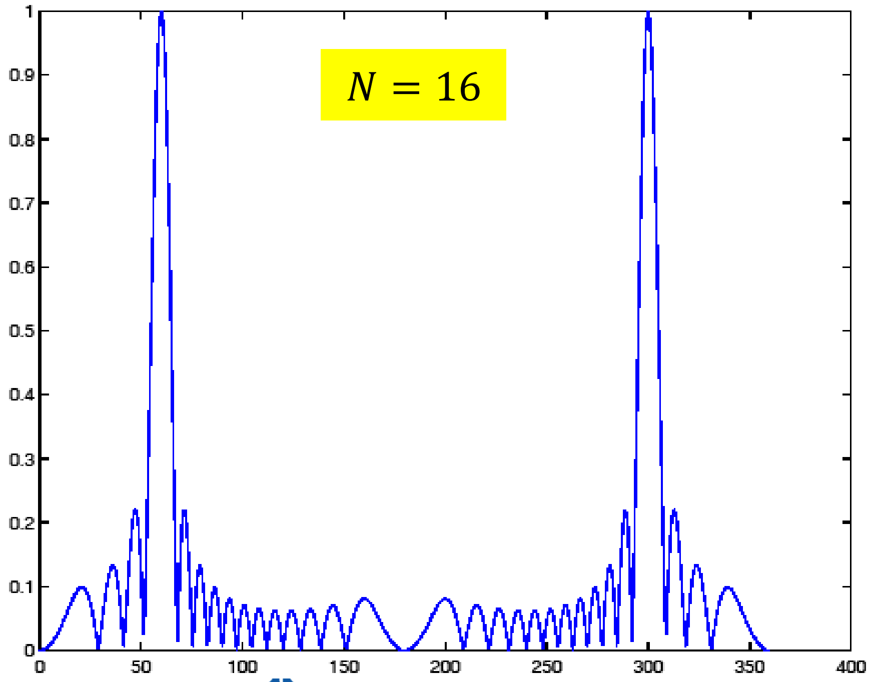
$N = 16$

Uniform Linear Arrays(10/17)

- Example (Beam pattern in θspace) (Cont.)

Signal direction $\tilde{\theta} = 60^{\circ}$; Inter-element spacing $d = \lambda/2$

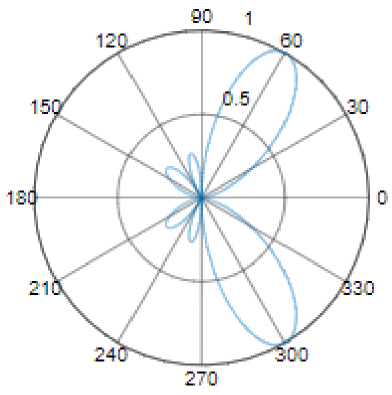


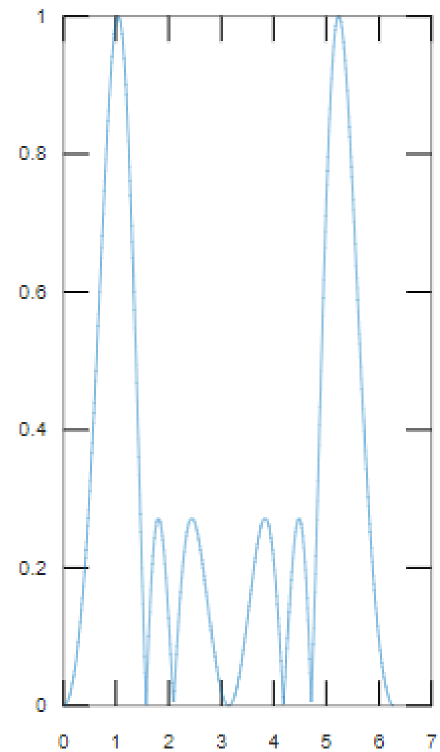


- https://octave-online.net/

$N = 4$

```
octave:6> theda = 0:0.01:2*pi; theda_tilde = 60*pi/180; N=4
N =  4
octave:7> B = 1/N .* sin(N .* pi * 1/2 .*(cos(theda) .- cos(theda_tilde))) ./ sin(pi.*1/2 .*(cos(theda) .- cos(theda_tilde)));
octave:8> subplot(1,2,1); polar(theda, abs(B)); subplot(1,2,2); plot(theda,abs(B))
```





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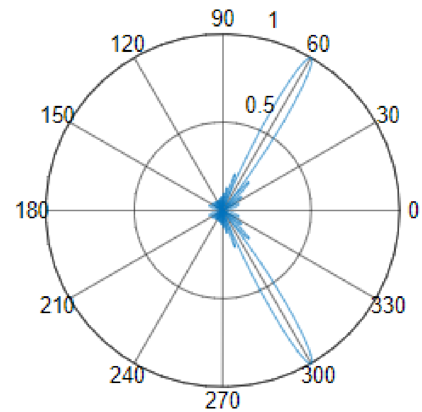
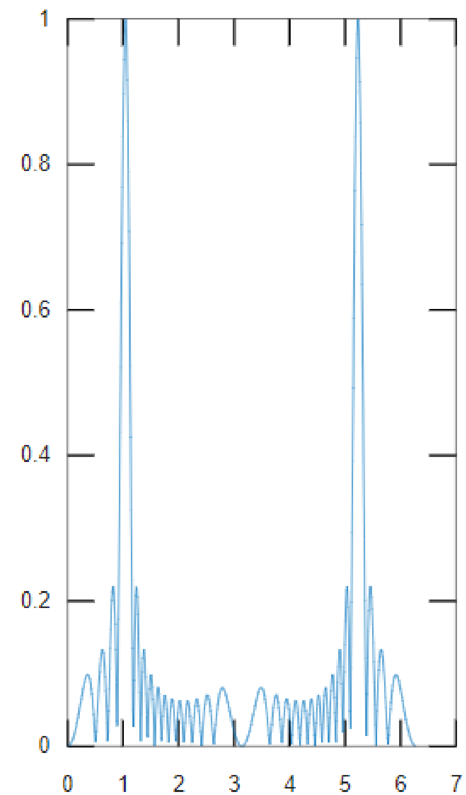
- https://octave-online.net/

$N = 16$

```
octave:9> theda = 0:0.01:2*pi; theda_tilde = 60*pi/180; N=16
N =  16
octave:10> B = 1/N .* sin(N .* pi * 1/2 .*(cos(theda) .- cos(theda_tilde))) ./ sin(pi.*1/2 .*(cos(theda) .- cos(theda_tilde)));
octave:11> subplot(1,2,1); polar(theda, abs(B)); subplot(1,2,2); plot(theda,abs(B))
```





Uniform Linear Arrays(11/17)

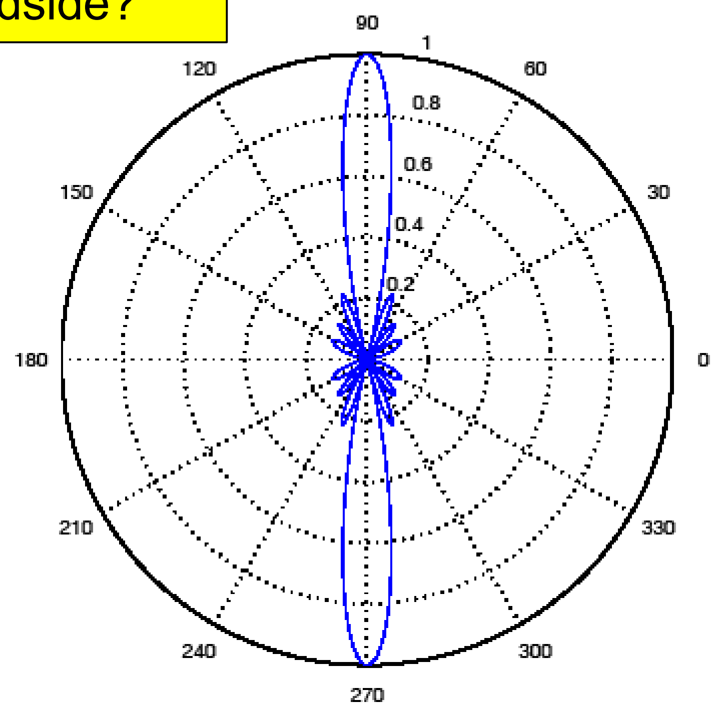
- Example (Beam pattern in θspace) (Cont.)
  - The previous plots are with the same wavelength, but with different number of antenna elements.
  - However, a shorter wavelength allows to use more antennas in the same physical space.
  - Therefore, if we consider the same aperture area for two systems working at two different carrier frequencies, the system with a shorter wavelength will allow beamforming with higher gains than the system with a longer wavelength.
  - Accordingly, the previous plots can also be interpreted as
    - With the same physical space, one's carrier frequency is four times the size of the other.
    - Thus the one with higher carrier frequency use more antennas (say $N = 16$) than the other ($N = 4$).

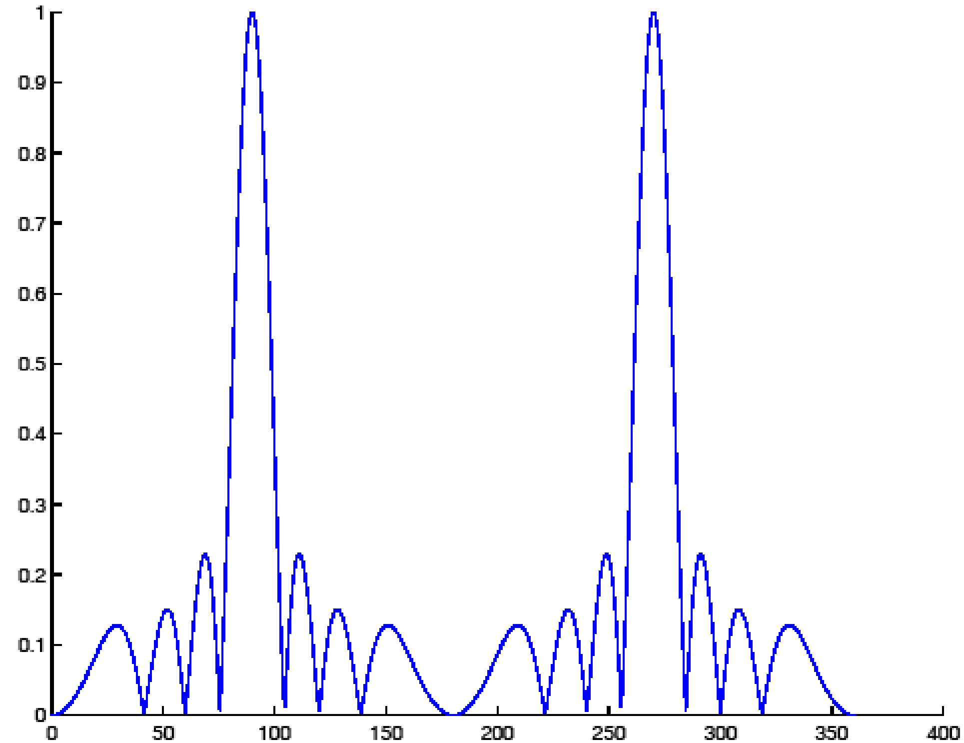
Uniform Linear Arrays (12/17)

- If the uniform weighting $\mathbf{w}^H = 1/N[1,1,\dots 1]$ is adopted, then the beam pattern in θ space is:

Think why the maximum gain occurs at broadside?

broadside





Uniform Linear Arrays (13/17)

- If the weight vector $\mathbf{w}$ is chosen from the vectors of DFT matrix $\mathbf{U}$ (DFT ***beambook***), we can put all the beam patterns associated with each weight vector into a polar plot.

DFT matrix

$$W = \left(\frac{\omega^{jk}}{\sqrt{N}} \right)_{j,k=0,\ldots,N-1}$$

$$\omega = e^{-2\pi j/N}$$

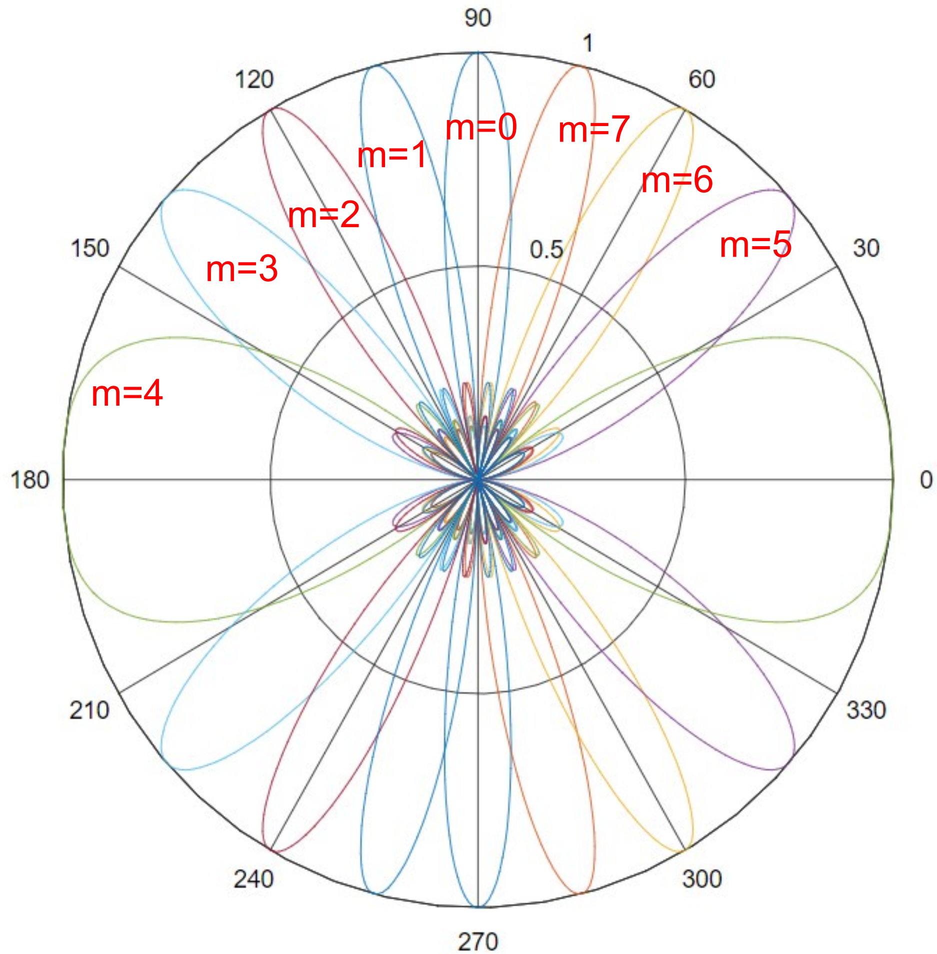
$$[\mathbf{U}]_{n,m} = \frac{1}{N} e^{\frac{-j2\pi nm}{N}}, \quad n,m = 0,1,\ldots,N-1$$

$$\mathbf{U} = \frac{1}{N} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{\frac{-j2\pi}{N}} & \cdots & e^{\frac{-j2\pi\cdot(N-1)}{N}} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & e^{\frac{-j2\pi(N-1)}{N}} & \cdots & e^{\frac{-j2\pi\cdot(N-1)\cdot(N-1)}{N}} \end{bmatrix}$$

Uniform Linear Arrays (14/17)

假設d為$\lambda/2$，N=8

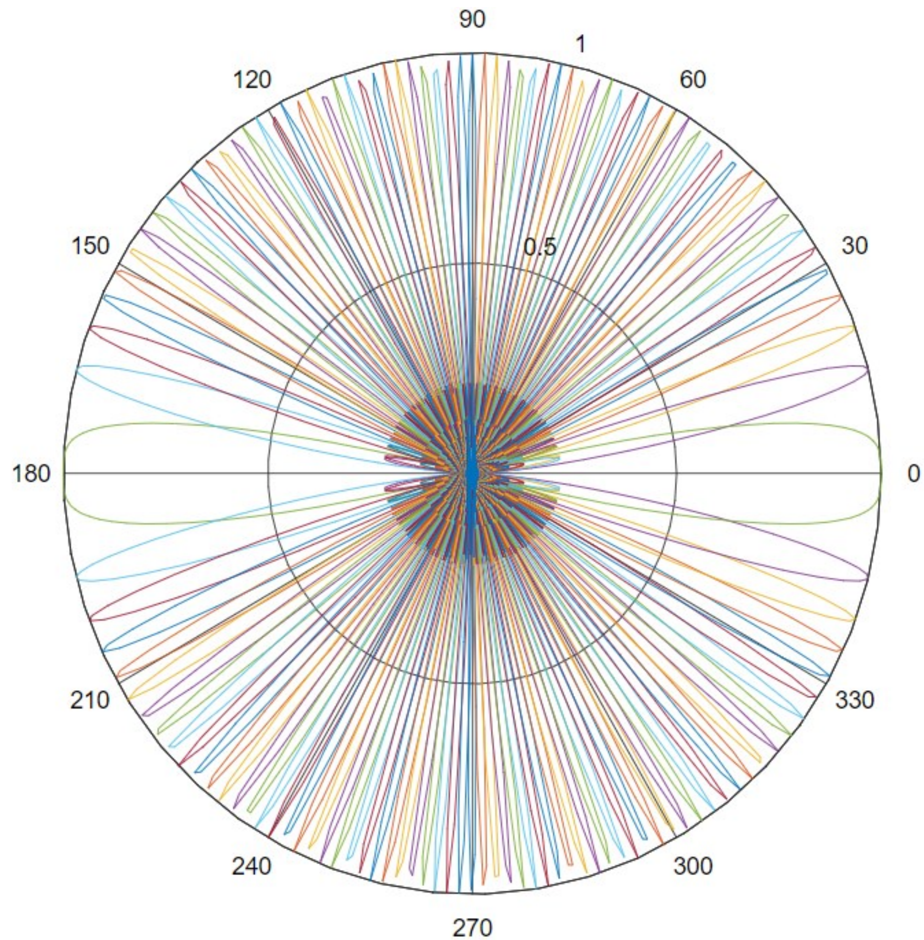
```
octave:6> theda = 0:0.01:2*pi; N=8;
v = e.^(j.*(0:N-1)'.*2.*pi.*1./2.*cos(theda));
U = 1./N .* e.^(-j.*2.*pi.*(0:N-1)'.*(0:N-1)./N);
B = U*v;
polar(theda,abs(B));
```



Uniform Linear Arrays (15/17)

假設d為$\lambda/2$，N=64

```
octave:11> theda = 0:0.01:2*pi; N=64;
v = e.^(j.*(0:N-1)'.*2.*pi.*1./2.*cos(theda));
U = 1./N .* e.^(-j.*2.*pi.*(0:N-1)'.*(0:N-1)./N);
B = U*v;
polar(theda,abs(B));
```



Uniform Linear Arrays (16/17)

- The beam patterns in different forms for this case:

[1, p.46]

$$B_{k_z} = \frac{1}{N} \frac{\sin\left(\frac{Nd}{2} k_z\right)}{\sin\left(\frac{d}{2} k_z\right)}, -\frac{2\pi}{\lambda} \le k_z \le \frac{2\pi}{\lambda}$$

$$B_{\theta} = \frac{1}{N} \frac{\sin\left(\frac{\pi N d}{\lambda} \cos\theta\right)}{\sin\left(\frac{\pi d}{\lambda} \cos\theta\right)}, 0 \le \theta \le \pi$$

$$B_{u} = \frac{1}{N} \frac{\sin\left(\frac{\pi N d}{\lambda} u\right)}{\sin\left(\frac{\pi d}{\lambda} u\right)}, -1 \le u \le 1$$

$$B_{\psi} = \frac{1}{N} \frac{\sin\left(\frac{N}{2} \psi\right)}{\sin\left(\frac{1}{2} \psi\right)}, -\frac{2\pi d}{\lambda} \le \psi \le \frac{2\pi d}{\lambda}$$

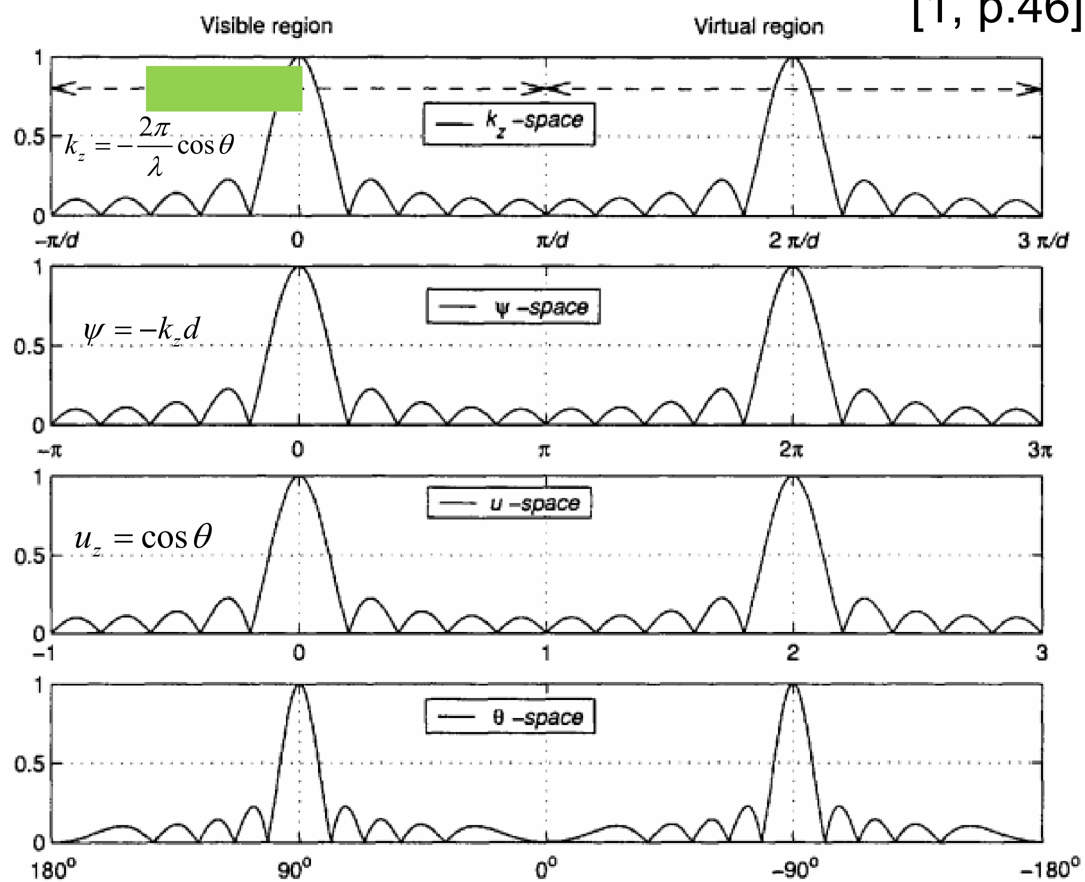
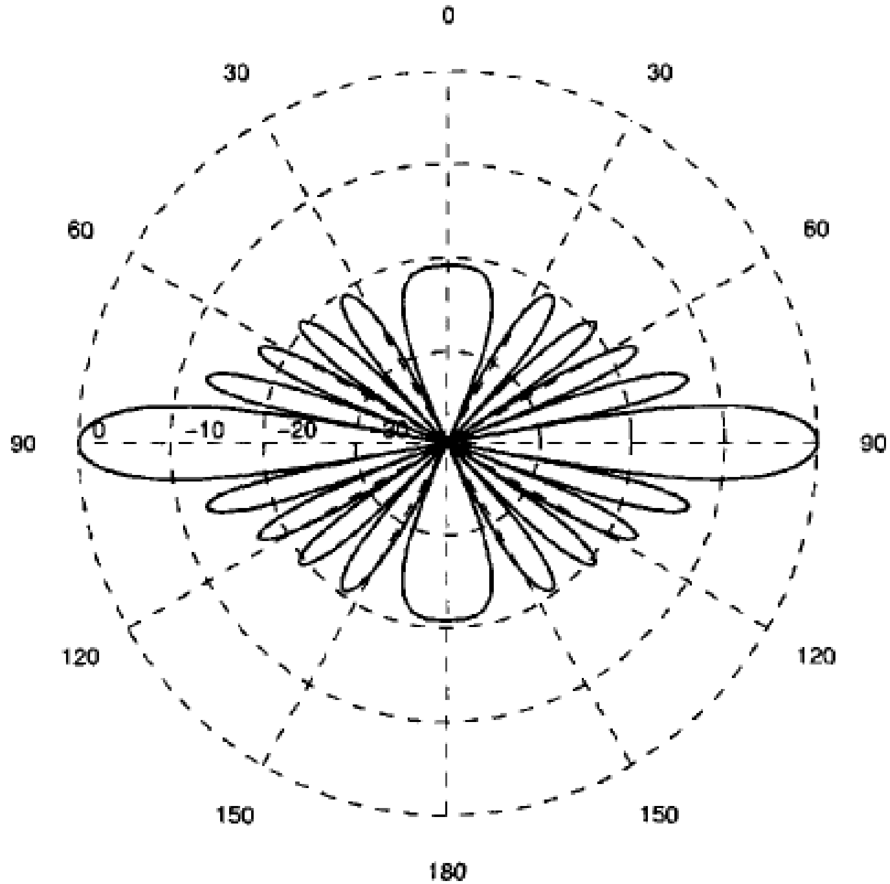


Figure 2.18 $|\Upsilon_\psi(\psi)|$ for a linear array with $d = \lambda/2$ and $N = 10$.

Uniform Linear Arrays (17/17)

Beam pattern for 10-element uniform array ($d = \lambda/2$)



[1, p.45]

Figure 2.17 Polar plot of $B_\theta(\theta)$. (in dB)

Outline

- Introduction
- Frequency-wavenumber Response and Beam Patterns
- Uniform Linear Arrays
- Array Performance Measures
- Array Steering

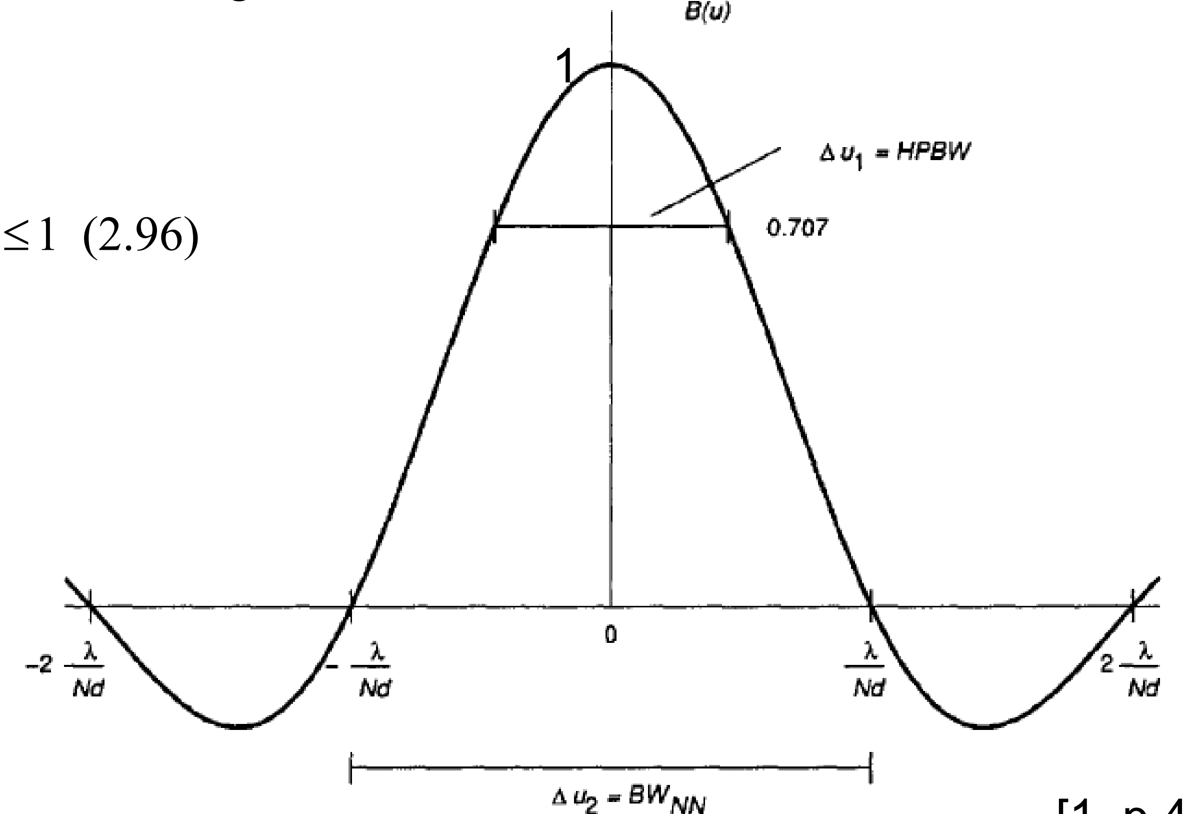
Beam Pattern Parameters

- 3-dB beamwidth (half-power beamwidth, HPBW)
- Distance to first null (twice this distance is *BWNN*)
- Rayleigh resolution limit
- Distance to first sidelobe and its height
- Location of remaining nulls
- Rate of decrease of sidelobes
- Grating lobes

Beam Pattern Parameters

- To illustrate the first two parameters, let us consider the beam pattern $B_u(u)$ near the origin.

$$B_u(u) = \frac{1}{N}\frac{\sin\left(\frac{\pi N d}{\lambda}u\right)}{\sin\left(\frac{\pi d}{\lambda}u\right)}, -1 \le u \le 1 \quad (2.96)$$



[1, p.47]

HPBW (3-dB Beamwidth)

- We can find the HPBW in u-space (Δu_1) by setting $|B_u(u)| = 1/\sqrt{2}$. Thus, $|B_u(u)|^2 = 0.5$.
- For $N \geq 10$, a good approximation is obtained by solving $\frac{\pi N d}{\lambda} u = 1.4$ (2.98).
- Then, $\frac{\Delta u_1}{2} = 1.4 \frac{\lambda}{\pi N d}$.
- $\rightarrow \Delta u_1 = 0.891 \frac{\lambda}{Nd}$

$$\bar{\theta} = \frac{\pi}{2} - \theta$$
$$u = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) = \sin(\bar{\theta})$$
$$\bar{\theta} = \sin^{-1}(u)$$
$$\rightarrow 0.5\Delta\bar{\theta} = \sin^{-1}(0.5\Delta u)$$
$$\rightarrow \Delta\bar{\theta} = 2\sin^{-1}(0.5\Delta u)$$

Table 2.2 HPBWs in Various Spaces

Space	Arbitrary d	$d = \lambda/2$
u	$0.891 \frac{\lambda}{Nd}$	$1.782 \frac{1}{N}$
$\bar{\theta}$	$2\sin^{-1}\left(0.446 \frac{\lambda}{Nd}\right)$	$2\sin^{-1}\left(0.891 \frac{1}{N}\right)$
small $\bar{\theta}$	$\simeq 0.891 \frac{\lambda}{Nd}$ radians $\simeq 51.05 \frac{\lambda}{Nd}$ degrees	$\simeq 1.782 \frac{1}{N}$ radians $\simeq 102.1 \frac{1}{N}$ degrees
ψ	$0.891 \frac{2\pi}{N}$	$0.891 \frac{2\pi}{N}$
k_z	$0.891 \frac{2\pi}{dN}$	$1.782 \frac{2\pi}{\lambda N}$

We define $\bar{\theta} = \pi/2 - \theta$ as the angle measured from broadside (see Figure 2.2).

[1, p.48]

[1]Problem 2.4.7

- In order to find the exact HPBW, we must solve

$$\left|B_u(u)\right|^2 = 0.5,$$

where $B_u(u)$ is given by (2.96).

  - (a)One approach to finding an approximate expression for the HPBW is to expand $|B_u(u)|^2$ in a second-order Taylor series around u =0. Use the expression for $B_u(u)$ given in (2.263) to simplify the derivatives. Compare your result to the result in (2.98).
  - (b)Use (2.96) to find the exact result as a function of N and compare the result with (2.98).

$$\frac{d\left|B_\psi(\psi)\right|^2}{d\psi}=-\frac{2}{N^2}\sum_{m=1}^{N-1}m(N-m)\sin m\psi$$

$$\frac{d^2\left|B_\psi(\psi)\right|^2}{d\psi}=-\frac{2}{N^2}\sum_{m=1}^{N-1}m^2(N-m)\cos m\psi$$

泰勒2階 $\left|B_\psi(\psi)\right|^2\approx\left|B_\psi(0)\right|^2+\psi\left.\frac{d\left|B_\psi(\psi)\right|^2}{d\psi}\right|_{\psi=0}+\frac{\psi^2}{2}\left.\frac{d^2\left|B_\psi(\psi)\right|^2}{d\psi}\right|_{\psi=0}+....$

$$=\frac{1}{\mathrm{N}}+\frac{2}{N^2}\sum_{m=1}^{N-1}(N-m)+\psi\left(-\frac{2}{N^2}\sum_{m=1}^{N-1}m(N-m)\sin m\psi\right)_{\psi=0}+\frac{\psi^2}{2}\left(-\frac{2}{N^2}\sum_{m=1}^{N-1}m^2(N-m)\cos m\psi\right)_{\psi=0}$$

$$=\frac{1}{\mathrm{N}}+\frac{2}{N^2}\left(\frac{(\mathrm{N\text{-}1+N\text{-}N+1})(\mathrm{N\text{-}1})}{2}\right)-\frac{\psi^2}{N^2}\left(\frac{N^4-N^2}{12}\right)$$

$$=\frac{1}{\mathrm{N}}+\frac{\mathrm{N(N\text{-}1)}}{N^2}-\psi^2\left(\frac{N^2-1}{12}\right)=1-\psi^2\left(\frac{N^2-1}{12}\right)$$

$$0.5=1-\psi^2\left(\frac{N^2-1}{12}\right)$$

$$0.5=\psi^2\left(\frac{N^2-1}{12}\right)$$

$$\frac{6}{N^2-1}=\psi^2$$

$$\sqrt{\frac{6}{N^2-1}}=\frac{2\pi d}{\lambda}u$$

$$\frac{\sqrt{6}}{2}\approx\frac{\pi dN}{\lambda}u$$

[1]Problem 2.6.6

- In order to find the directivity of a uniformly weighted linear array that is pointed at ψ_T , it is convenient to rewrite the beam pattern
- (a)Show that

$$B_\psi(\psi) = \frac{1}{N}\left\{1 + 2\sum_{m=1}^{\frac{N-1}{2}} \cos m\psi\right\}, \text{ N odd}, \quad (2.260)$$

and

$$B_\psi(\psi) = \frac{1}{N}\left\{2\sum_{m=0}^{\frac{N}{2}-1} \cos(\mathrm{m} - \frac{N-1}{2})\psi\right\}, \text{ N even}, \quad (2.261)$$

where

$$\psi = \frac{2\pi d}{\lambda}\cos\theta - \psi_T = \frac{2\pi d}{\lambda}\cos\theta = \frac{2\pi d}{\lambda}\cos\theta_T \quad (2.262)$$

- (b)We then written

$$\left|B_\psi(\psi)\right|^2 = \left|\frac{\sin\frac{N\psi}{2}}{N\sin\frac{\psi}{2}}\right|^2 = \frac{1}{N} + \frac{1}{N^2}\sum_{m=1}^{N-1}(N-m)\cos m\psi, \quad (2.263)$$

H.W.
- Verify this expression for N=2, 3, 4, and 5

Null-to-null Beamwidth (BW_{NN})

- The nulls of the pattern occur when the numerator of $B_u(u)$ is zero and the denominator is non-zero:

$$\frac{1}{N}\frac{\sin\left(\frac{\pi N d}{\lambda}u\right)}{\sin\left(\frac{\pi d}{\lambda}u\right)}=0$$

$$\rightarrow \frac{\pi N d}{\lambda}u = m\pi, \quad m = 1, 2, \cdots$$

$$\Rightarrow u = \frac{\lambda}{Nd}m$$

$$\rightarrow \frac{\pi d}{\lambda}u \neq m\pi, \quad m = 1, 2, \cdots$$

$$\Rightarrow u \neq \frac{\lambda}{d}m$$

$$\rightarrow BW_{NN} = \Delta u_2 = 2\frac{\lambda}{Nd}$$

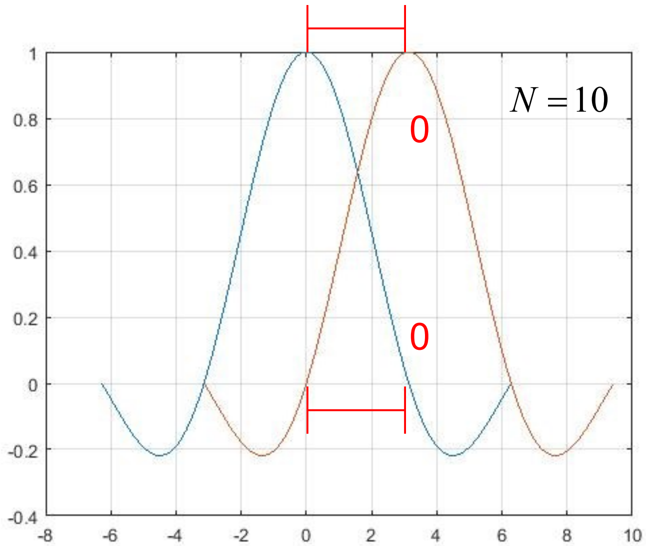
[1, p.49]

Table 2.3 BW_{NN} **in Various Spaces**

Space	Arbitrary d	$d=\lambda/2$
u	$2\frac{\lambda}{Nd}$	$\frac{4}{N}$
$\bar{\theta}$	$2\sin^{-1}\left(\frac{\lambda}{Nd}\right)$	$2\sin^{-1}\left(\frac{2}{N}\right)$
small $\bar{\theta}$	$\simeq 2\frac{\lambda}{Nd}$ radians	$\simeq \frac{4}{N}$ radians
ψ	$\frac{4\pi}{N}$	$\frac{4\pi}{N}$
k_z	$\frac{4\pi}{dN}$	$\frac{8\pi}{\lambda N}$

Rayleigh Resolution Limit

- One-half of BW_{NN} is referred to as the **Rayleigh resolution limit**. This quantity provides a measure of the ability of the array to resolve two different plane waves.
- Two plane waves are considered resolvable if the peak of the second beam pattern lies at or outside of the null of the first beam pattern (separation $\geq \Delta u_2/2$).



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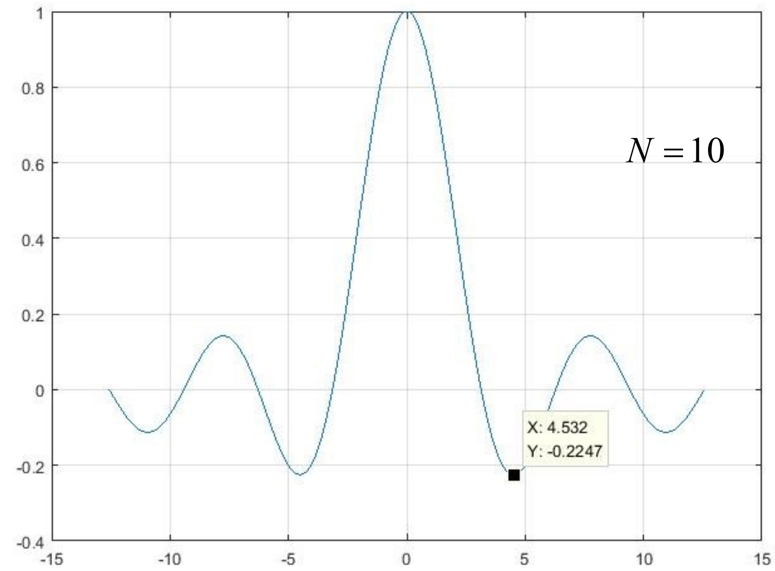
Distance to the First Sidelobe and Its Height

- The maxima of the sidelobes occurs when the numerator of beam pattern is a maximur



$$B_u(u)=\frac{1}{N}\frac{\sin\left(\frac{\pi Nd}{\lambda}u\right)}{\sin\left(\frac{\pi d}{\lambda}u\right)},-1\le u\le 1$$

$$\to\frac{\pi Nd}{\lambda}u=\pm\left(\frac{\pi}{2}+m\pi\right)=\pm\frac{\pi}{2}(2m+1),\quad m=1,2,\cdots$$

$$\to u=\pm\frac{\lambda}{2Nd}(2m+1),m=1,2,\cdots$$

$\to$ Then the peak of the first sidelobe occurs at

$$u=\pm\frac{3\lambda}{2Nd}\quad(\text{as } m=1)$$

$\to$ The maxima of the first sidelobe is

$$\frac{1}{N}\frac{\sin\left(\frac{\pi Nd}{\lambda}u\right)}{\sin\left(\frac{\pi d}{\lambda}u\right)}=\frac{1}{N}\frac{\sin\left(\frac{3\pi}{2}\right)}{\sin\left(\frac{3\pi}{2N}\right)}$$

$$m=1,N=10\to\frac{(2m+1)\pi}{2}=\frac{3\pi}{2}=4.71$$

$$\frac{\pi Nd}{\lambda}u$$

$$\to\frac{1}{10}\frac{\sin\left(\frac{3\pi}{2}\right)}{\sin\left(\frac{3\pi}{20}\right)}=-0.2203$$

Grating Lobes (1/2)

- A grating lobe is a lobe of the same height as the main lobe.
- Grating lobes occur when both the numerator and denominator of beam pattern equal zero.

$$B_\psi = \frac{1}{N}\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)} \rightarrow \frac{\psi}{2} = m\cdot\pi \rightarrow \psi = m\cdot 2\pi = m\frac{2\pi d}{\lambda}u \rightarrow u = m\cdot\frac{\lambda}{d}, \quad m = 1, 2, \ldots$$

$$\psi = \frac{2\pi d}{\lambda}u, 2\pi m = \frac{2\pi d}{\lambda}u$$

- To avoid the occurrence of grating lobes, it should be $d \le \frac{\lambda}{2}$.

Visible region in u-space: $-1 \le u \le 1$

$$\rightarrow \left|u_m - u_{m+1}\right| = \frac{\lambda}{d} \le 2 \ \rightarrow \ d \le \frac{\lambda}{2}$$

如果移動角度也只有一個main lobe
希望visible region只有一個main lobe

We refer to a ULA with $d = \lambda/2$ as a **standard linear array**.

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Grating Lobes (2/2)

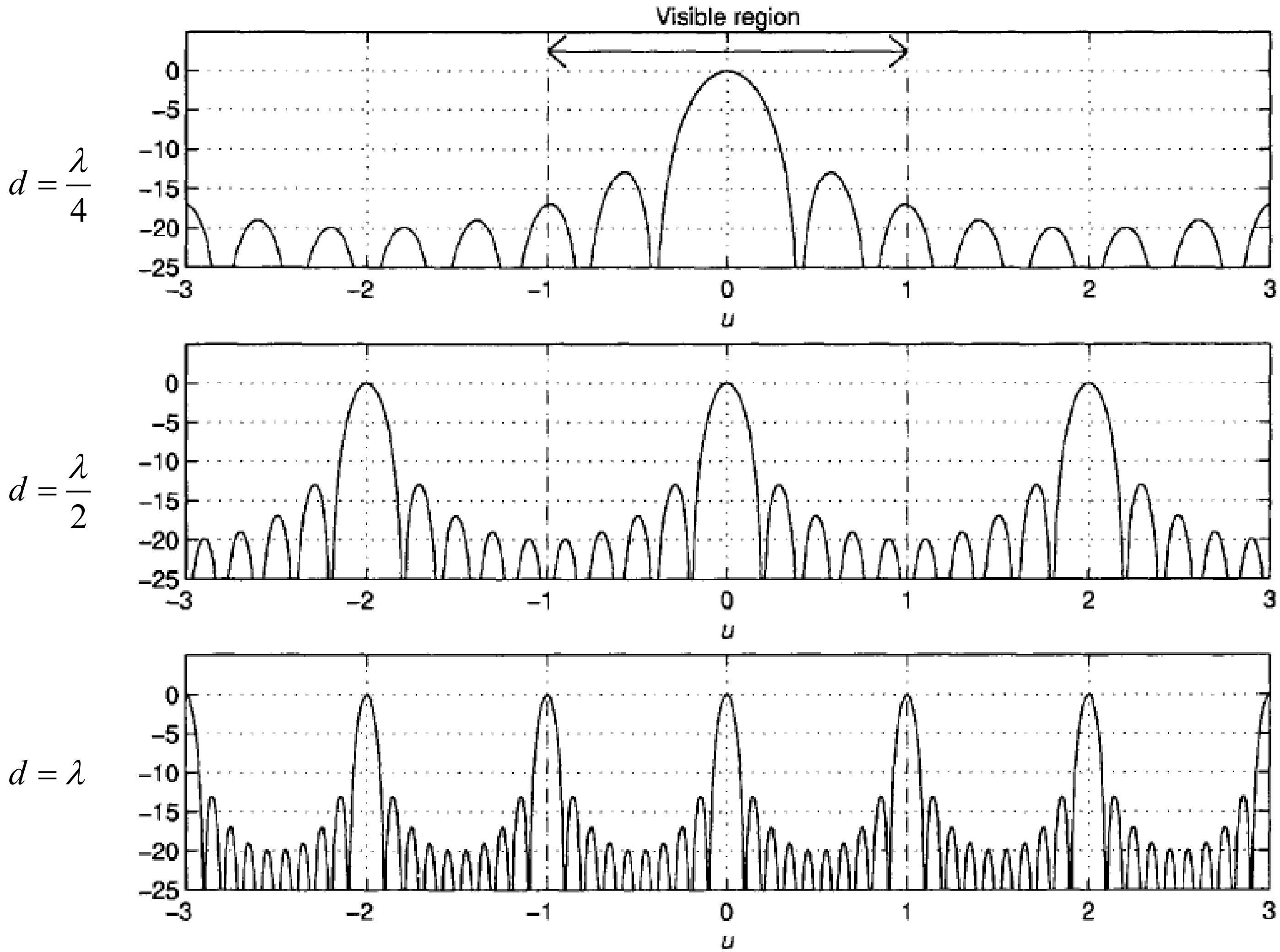


Figure 2.20 Effect of element spacing on beam pattern: (a) $d = \lambda/4$; (b) $d = \lambda/2$; (c) $d = \lambda$.

The problem of grating lobes is identical to the problem of aliasing in time series analysis, which occurs when we ***undersample*** the time domain waveform.

[1, p.51]

Outline

- Introduction
- Frequency-wavenumber Response and Beam Patterns
- Uniform Linear Arrays
- Array Performance Measures
- Array Steering

Array Steering (1/9)

- The maximum response of an array occurs at broadside, or $k_z = 0$.(when w=1 for all element)

意思是每個antenna element的weight要一樣，
就是broad side 直直往前打

- In most applications we want to be able to position (or steer) this response to an arbitrary wavenumber or direction. This is called *array steering*.
- Array steering cause the frequency-wavenumber function (say in *u*-space) to shift in *u*-space.
- Note that this shift can cause grating lobes to move into the visible region.

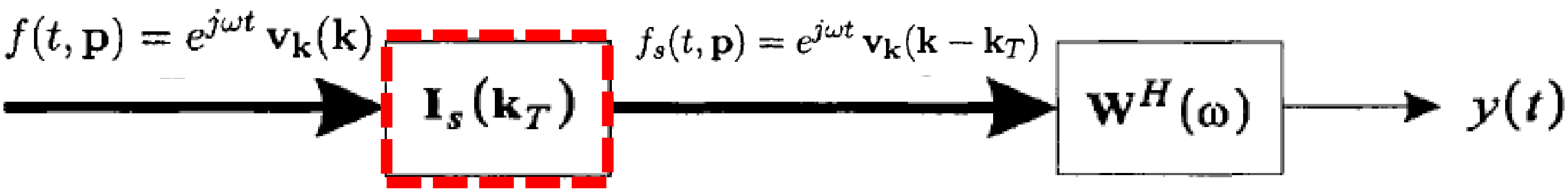
Array Steering (2/9)

- There are two ways to accomplish array steering: mechanical and electronic steering.
- **Mechanical steering** is to change the location of the sensors so that the axis is perpendicular to the desired steering direction.
  - Often mechanical steering is not possible because of either the large physical dimensions of an array when operating with long wavelength signals or the need to re-calibrate sensors when they are moved.
- **Electronic steering** is to introduce time delays (or in the narrowband case, phase shifts) to steer the main response axis(MRA) of an array.
  - With the advances in very high speed signal processors, electronic steering is being used much more extensively in array processing, not only because of the restrictions of mechanical steering but also because of its flexibility and its ability to change the response function rapidly.

Array Steering (3/9)

- Consider the steering processor $\mathbf{I}_s\,(\mathbf{k}_T)$ to steer the array in a specific direction $\mathbf{k}_T$, which is the target wavenumber or **steering direction**.
- The output would be aligned if $\mathbf{k} = \mathbf{k}_T$. We accomplish this with an $N \times N$ diagonal steering matrix

$$\mathbf{I}_s(\mathbf{k}_T) \triangleq \begin{bmatrix} e^{j\mathbf{k}_T^T\mathbf{p}_0} & 0 & \cdots & 0 \\ 0 & e^{j\mathbf{k}_T^T\mathbf{p}_1} & \cdots & 0 \\ 0 & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & e^{j\mathbf{k}_T^T\mathbf{p}_{N-1}} \end{bmatrix} \cdot \qquad \mathbf{v}_{\mathbf{k}}(\mathbf{k}) = \begin{bmatrix} e^{-j\mathbf{k}^T\mathbf{p}_0} \\ e^{-j\mathbf{k}^T\mathbf{p}_1} \\ \vdots \\ e^{-j\mathbf{k}^T\mathbf{p}_{N-1}} \end{bmatrix}$$



Now, the weight vector **w** can be MRC, etc.

[1, p.52]

Array Steering (4/9)

- The overall frequency-wavenumber response is

$$\Upsilon(\omega,\mathbf{k}|\mathbf{k}_T)=\mathbf{H}^T(\omega)\mathbf{v}_{\mathbf{k}}(\mathbf{k}-\mathbf{k}_T)=\mathbf{w}^H(\omega)\mathbf{v}_{\mathbf{k}}(\mathbf{k}-\mathbf{k}_T).$$

- When we use uniform amplitude weighting $\mathbf{w}^H=(1/N)\,[1,1,\ldots 1]$, the beam pattern is

$$B_c(\mathbf{k}:\mathbf{k}_T)=\mathbf{w}^H\mathbf{v}_{\mathbf{k}}(\mathbf{k}-\mathbf{k}_T)=\frac{1}{N}\sum_{n=0}^{N-1}e^{-j(\mathbf{k}-\mathbf{k}_T)^T\mathbf{p}_n}$$

$$=\frac{1}{N}\mathbf{v}_{\mathbf{k}}^H(\mathbf{k}_T)\mathbf{v}_{\mathbf{k}}(\mathbf{k}).$$

- We refer to $B_c(\mathbf{k}:\mathbf{k}_T)$ as the **conventional beam pattern**.

Array Steering (5/9)

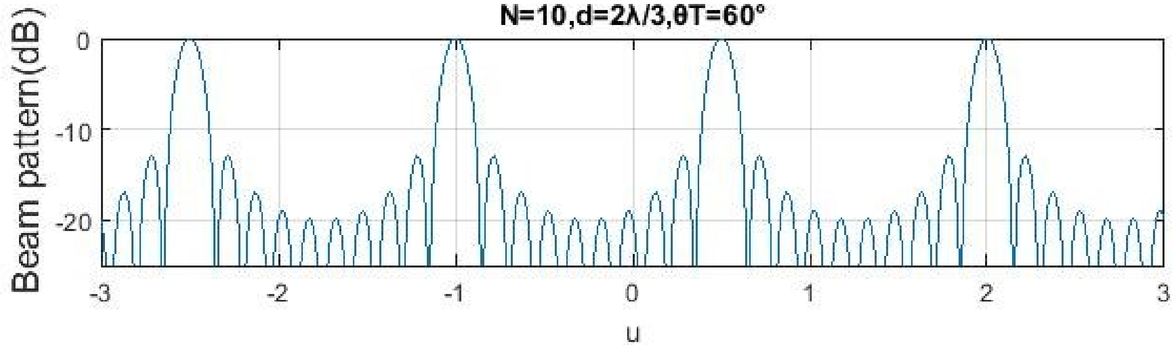
- For a ULA, the conventional beam pattern can be written

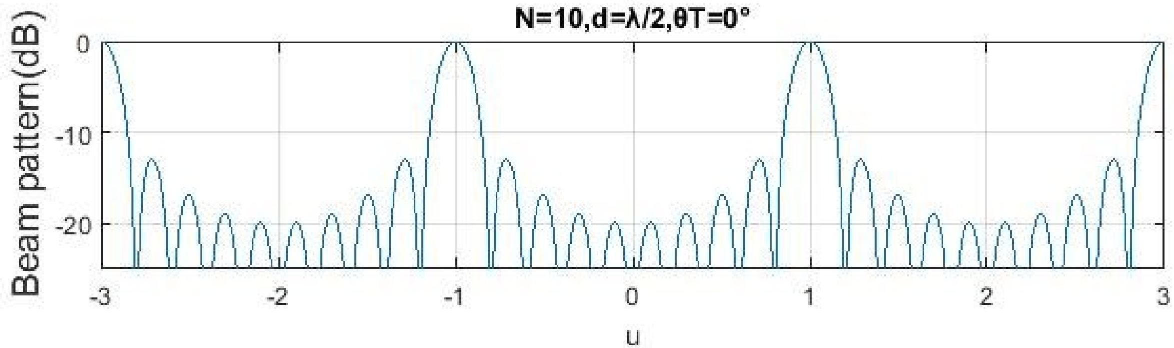
$$B_{\psi}\left(\psi:\psi_T\right)=\frac{1}{N}\mathbf{v}_{\psi}^{H}\left(\psi_T\right)\mathbf{v}_{\psi}\left(\psi\right)=\frac{1}{N}\frac{\sin\left(\frac{N}{2}\left(\psi-\psi_T\right)\right)}{\sin\left(\frac{1}{2}\left(\psi-\psi_T\right)\right)}$$

$$B_{u}\left(u:u_T\right)=\frac{1}{N}\mathbf{v}_{u}^{H}\left(u_T\right)\mathbf{v}_{u}\left(u\right)=\frac{1}{N}\frac{\sin\left(\frac{\pi Nd}{\lambda}\left(u-u_T\right)\right)}{\sin\left(\frac{\pi d}{\lambda}\left(u-u_T\right)\right)}$$

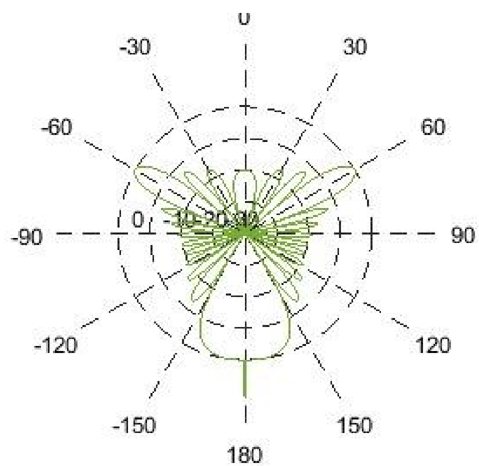
$$B_{\theta}\left(\theta:\theta_T\right)=\frac{1}{N}\mathbf{v}_{\theta}^{H}\left(\theta_T\right)\mathbf{v}_{\theta}\left(\theta\right)=\frac{1}{N}\frac{\sin\left(\frac{\pi Nd}{\lambda}\left(\cos\theta-\cos\theta_T\right)\right)}{\sin\left(\frac{\pi d}{\lambda}\left(\cos\theta-\cos\theta_T\right)\right)}$$

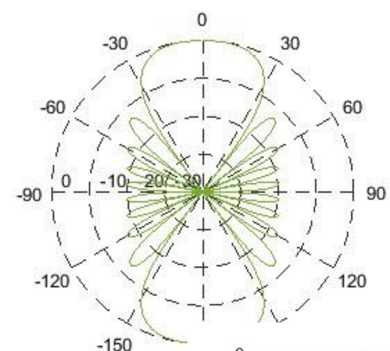
The expressions in ψ-space and *u*-space correspond to shifts in the pattern, but its shape is not changed. This property of shifting without distortion is one of many advantages of working in ψ-space and *u*-space.

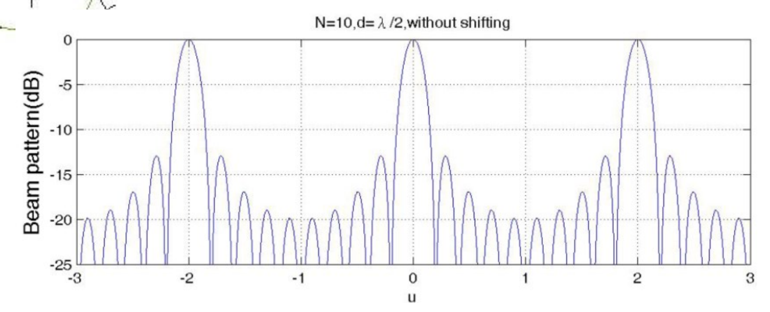




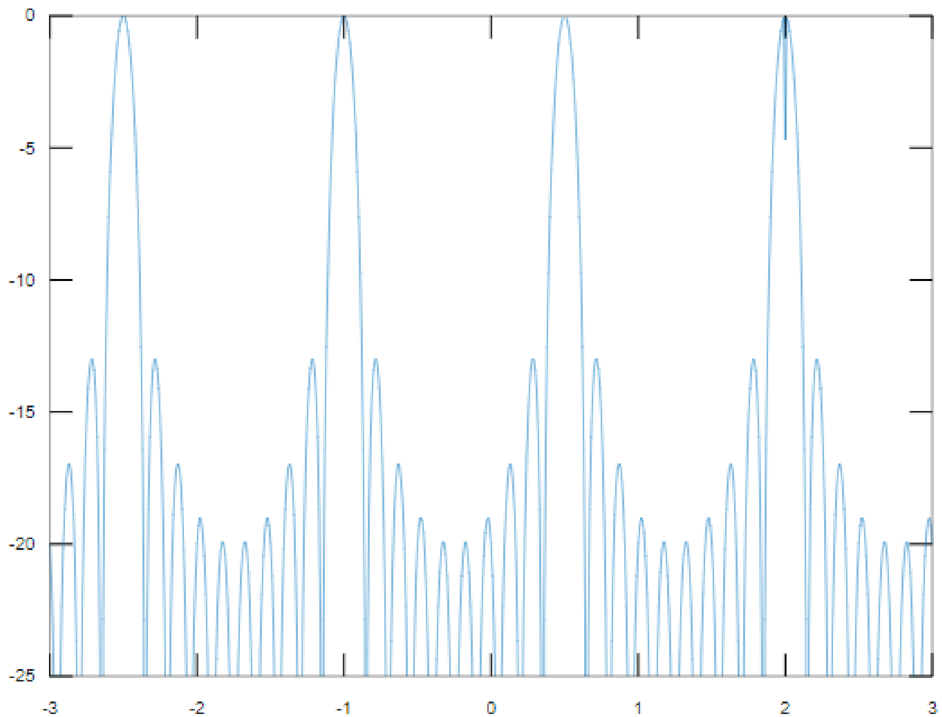
Note that this shift can cause grating lobes to move into the visible region.

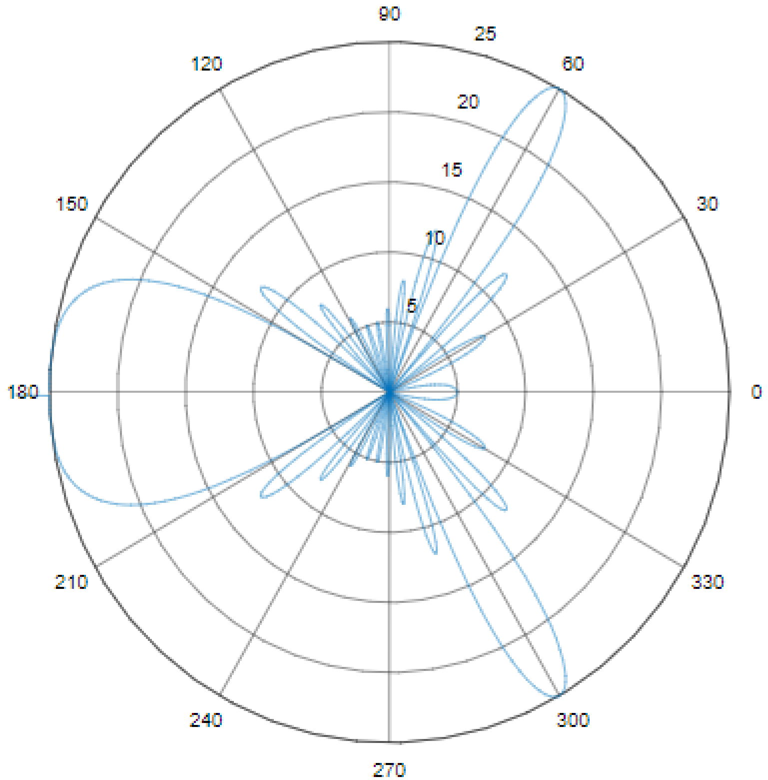




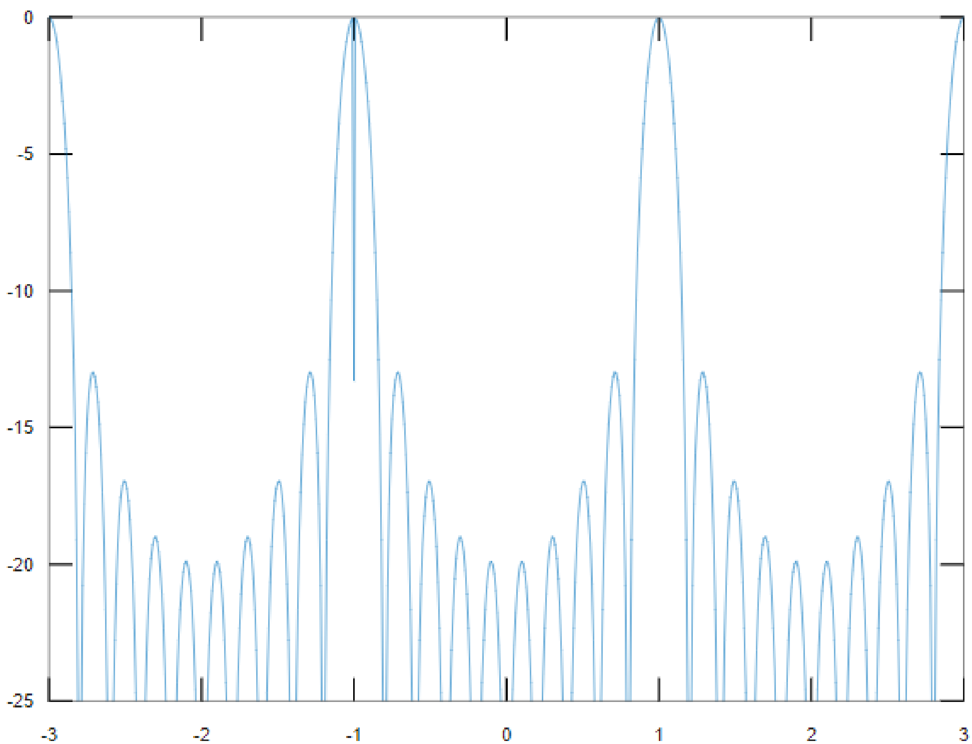


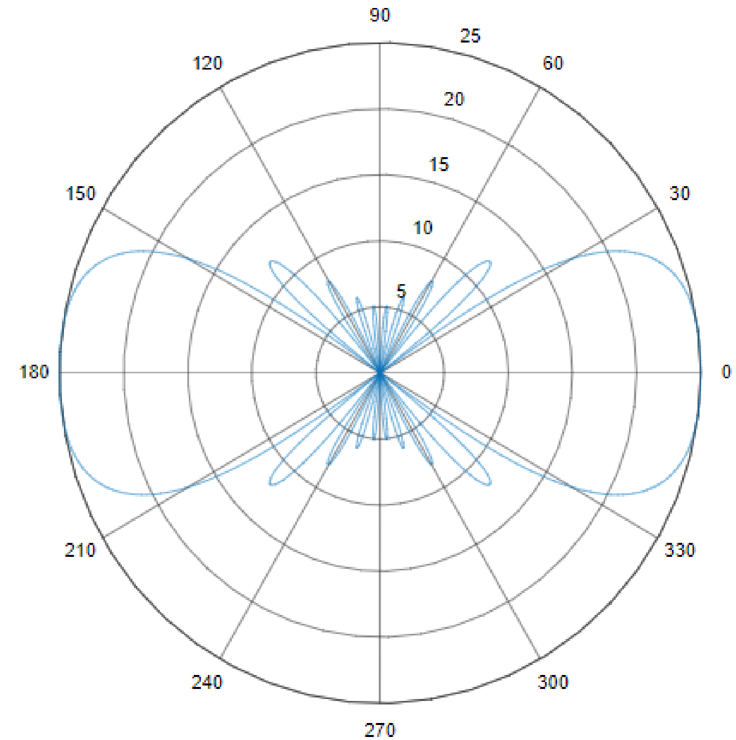
```
octave:19> u = -3:0.01:3; N=10; d=2/3; uT=cosd(60);
B = 1/N .*sin(N .* pi .* d .*(u-uT))./sin(pi .* d .*(u-uT));
theda = -pi:0.01:pi;

BB = 1/N .* sin(N .* pi .* d .* (cos(theda)-uT))./sin(pi .* d .*(cos(theda)-uT));
plot(u,10.*log10(abs(B).^2)); ylim([-25,0]);
polar(theda, 25+max(10.*log10(abs(BB).^2),-25)); set (gca, 'rtick', [0:5:25]);
```





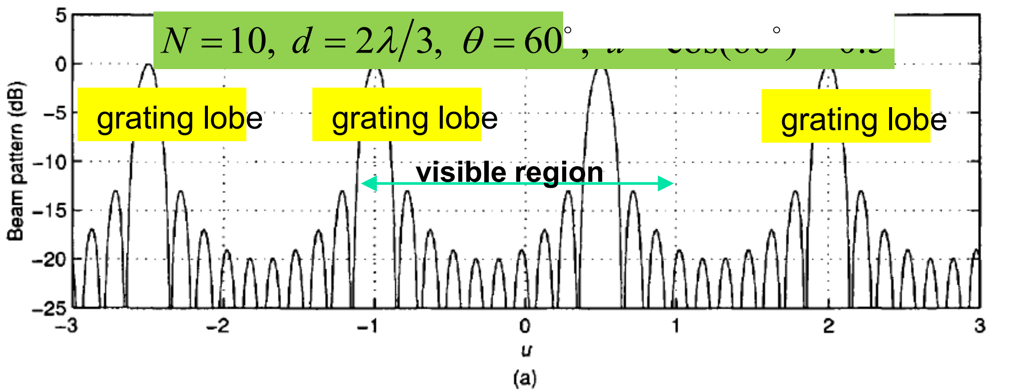
```
octave:31> u = -3:0.01:3; N=10; d=1/2; uT=cosd(0);
B = 1/N .*sin(N .* pi .* d .*(u-uT))./sin(pi .* d .*(u-uT));
theda = -pi:0.01:pi;

BB = 1/N .* sin(N .* pi .* d .* (cos(theda)-uT))./sin(pi .* d .*(cos(theda)-uT));
plot(u,10.*log10(abs(B).^2)); ylim([-25,0]);
polar(theda, 25+max(10.*log10(abs(BB).^2),-25)); set (gca, 'rtick', [0:5:25]);
```

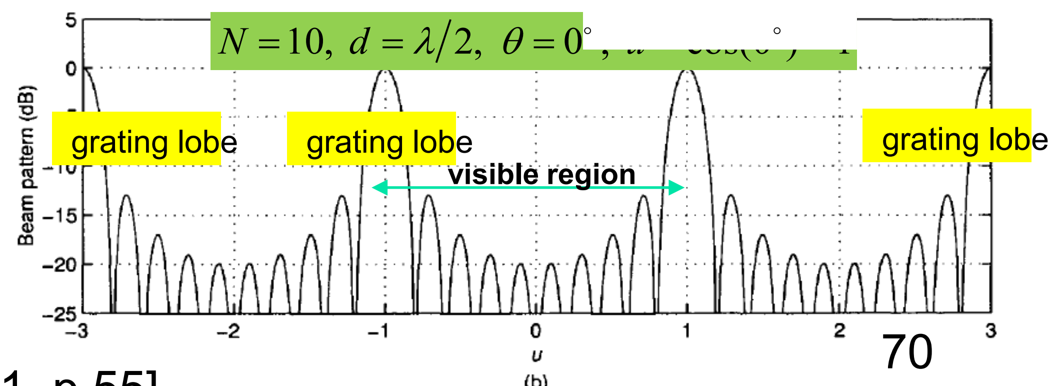




Array Steering (6/9)

- As we steer the array so that the main response axis is aimed at θ, the u-space beam pattern shifts so that the center peak is at $u = \cos(\theta)$. This shift causes the grating lobes to move into the visible region.





[1, p.55]

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Array Steering (7/9)

- In order to avoid a grating lobe from moving into the visible region, in general we require

$$\frac{d}{\lambda} \leq \frac{1}{1+\left|\cos\theta_{\max}\right|} = \frac{1}{1+\left|\sin\bar{\theta}_{\max}\right|},$$

$$\bar{\theta}_{\max} = \frac{\pi}{2} - \theta_{\max} \text{ is the angle measured from broadside.}$$

  - $\theta_{\max}$ is the maximum angle to which the array will be required to steer.

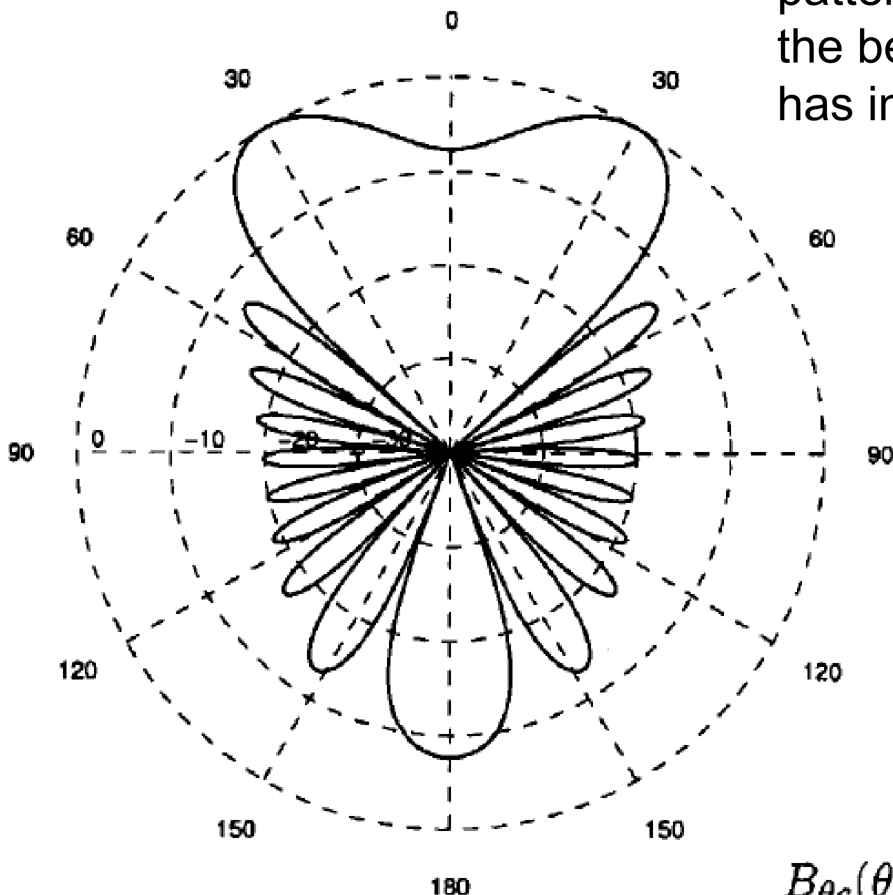
  - This result follows from calculating the location of the first grating lobe as a function of d/λ with $\theta_T = \theta_{\max}$. Thus, the restriction is $d \leq \lambda/2$, if the array is required to steer $0 \leq \theta \leq 180^{\circ}$.

Array Steering (8/9)

- The behavior in ψ-space and u-space is useful. However, the signals originate in a (θ, ϕ) space, and we need to understand the behavior in that space.
- In θ-space (i.e., angle space),

$$B_\theta(\theta:\theta_T) = \frac{1}{N}\mathbf{v}_\theta^H(\theta_T)\mathbf{v}_\theta(\theta) = \frac{1}{N}\frac{\sin\left(\frac{\pi N d}{\lambda}(\cos\theta - \cos\theta_T)\right)}{\sin\left(\frac{\pi d}{\lambda}(\cos\theta - \cos\theta_T)\right)}.$$

Comparing this pattern with the pattern in Fig. 2.17, we see that the beamwidth of the main lobe has increased.



Conventional beam pattern after shifting 60° from broadside

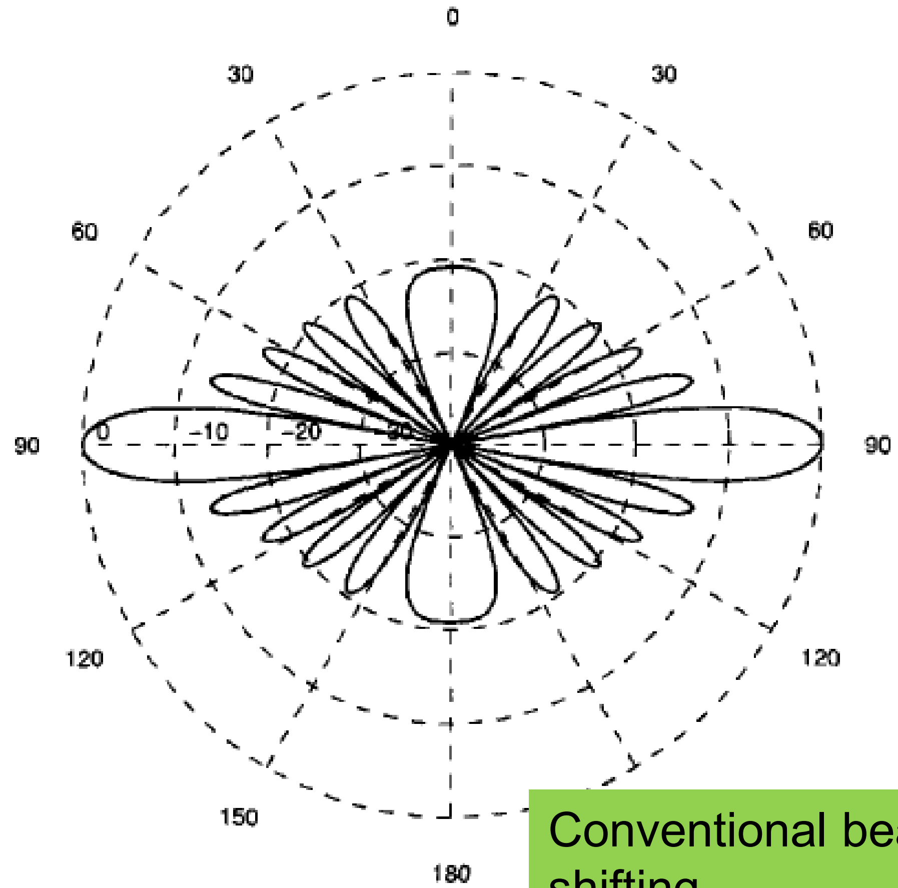
$B_{\theta c}(\theta : \theta_T)$ in θ-space

Figure 2.23 Beam pattern for 10-element uniform array ($d = \lambda/2$) scanned to 30° (60° from broadside).

[1, p.56]

Beam pattern for 10-element uniform array ($d = \lambda/2$)

H.W.



Conventional beam pattern without shifting

Figure 2.17 Polar plot of $B_\theta(\theta)$. (in dB)

[1, p.45]

$$\Delta u_1 = 0.891\frac{\lambda}{Nd}. \quad (2.100)$$

Array Steering (9/9)

$$B_{\theta c}(\theta : \theta_T) = \frac{1}{N}\frac{\sin[\frac{\pi Nd}{\lambda}(\cos\theta - \cos\theta_T)]}{\sin[\frac{\pi d}{\lambda}(\cos\theta - \cos\theta_T)]}. \quad (2.131)$$

- To investigate the behavior of the HPBW in θ-space, we use (2.131) and (2.100). The right and left half-power point in *u*-space are respectively:

$$u_R = u_T + 0.450\frac{\lambda}{Nd}, \quad u_L = u_T - 0.450\frac{\lambda}{Nd}.$$

- Change to θ-space (θ_R corresponds to $u_L$and θ_L corresponds to u_R)

$$\cos\theta_R = \cos\theta_T - 0.450\frac{\lambda}{Nd}$$

$$\cos\theta_L = \cos\theta_T + 0.450\frac{\lambda}{Nd}$$

$$\theta_H = \theta_R - \theta_L = \cos^{-1}\left(\cos\theta_T - 0.450\frac{\lambda}{Nd}\right) - \cos^{-1}\left(\cos\theta_T + 0.450\frac{\lambda}{Nd}\right)$$

不同角度時，HPBW in θ-space的寬度會不一樣(越靠Z軸越胖)

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參考資料

[1]Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory, Harry L. Van Trees, April 2004 – Chapter 2

Thank you